SZEGŐ KERNEL ASYMPTOTICS AND KODAIRA EMBEDDING THEOREMS OF LEVI-FLAT CR MANIFOLDS

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ABSTRACT. Let X be an orientable compact Levi-flat CR manifold and let L be a positive CR complex line bundle over X. We prove that certain microlocal conjugations of the associated Szegő kernel admits an asymptotic expansion with respect to high powers of L. As an application, we give a Szegő kernel proof of the Kodaira type embedding theorem on Levi-flat CR manifolds due to Ohsawa and Sibony.

CONTENTS

1. INTRODUCTION AND STATEMENT OF THE MAIN RESULTS

The problem of global embedding CR manifolds is prominent in areas such as complex analysis, partial differential equations and differential geometry. A general result is the CR embedding of strictly pseudoconvex compact CR manifolds of dimension greater than five, due to Boutet de Monvel [2].

For CR manifolds which are not strictly pseudoconvex, the idea of embedding CR manifolds by means of CR sections of tensor powers L^k of a positive CR line bundle $L \to X$ was considered in [12, 13, 14, 20, 25]. This was of course inspired by Kodaira's embedding theorem.

One way to attack this problem is to produce CR sections by projecting appropriate smooth sections to the space of CR sections. So it is crucial to understand the large k behaviour of the Szegő projection Π_k, i. e. the orthogonal projection on space $H^0_b(X, L^k)$ of CR sections, and of

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its distributional kernel, the Szegő kernel. To study the Szegő projection it is convenient to link it to a parametrix of the $\overline{\partial}_b$-Laplacian on $(0,1)$-forms (called Kohn Laplacian). This is also the method used in [2], where the parametrix turns out to be a pseudodifferential operator of order $1/2$.

In [14], we established analogues of the holomorphic Morse inequalities of Demailly [5, 19] for CR manifolds and we deduced that the space $H^0_b(X, L^k)$ is large under the assumption that the curvature of the line bundle is adapted to the Levi form. In [12], the first author introduced a microlocal cut-off function technique and could remove the assumptions linking the curvatures of the line bundle and the Levi form under rigidity conditions on X and the line bundle. Moreover, in [13], the first author established partial Szegő kernel asymptotic expansions and Kodaira embedding theorems on CR manifolds with transversal CR S^1 actions.

All these developments need the assumptions that either the curvature of the line bundle is adapted to the Levi form or rigidity conditions on X and the line bundle. The difficulty of this kind of problem comes from the presence of positive eigenvalues of the curvature of the line bundle and negative eigenvalues of the Levi form of X. Thus, it is very interesting to consider Levi-flat CR manifolds. In this case, the eigenvalues of the Levi form are zero and we will show that it is possible to remove the assumptions linking the curvatures of the line bundle and the Levi form or the rigidity conditions on X and the line bundle.

Actually, Ohsawa and Sibony [25], cf. also [24], constructed a CR projective embedding of class $\mathscr{C}^\kappa$ for any $\kappa \in \mathbb{N}$ of a Levi-flat CR manifold by using $\overline{\partial}$-estimates. A natural question is whether we can improve the regularity to $\kappa = \infty$. Adachi [1] showed that the answer is no, in general. The analytic difficulty of this problem comes from the fact that the Kohn Laplacian is not hypoelliptic on Levi flat manifolds. Hypoellipticity and subelliptic estimates are used on CR manifolds with non-degenerate Levi form in order to find parametrices of the Kohn Laplacian and establish the Hodge decomposition, e. g. [2, 4, 15, 16]. Moreover, the Szegő projection Π_k is not a Fourier integral operator in our case.

In this paper, we establish a semiclassical Hodge decomposition for the the Kohn Laplacian acting on powers L^k as $k \to \infty$ and we show that the composition $\Pi_k \circ \mathcal{A}_k$ of Π_k with an appropriate pseudodifferential operator $\mathcal{A}_k$ is a semiclassical Fourier integral operator, admitting an asymptotic expansion in k (see Theorem 1.3). From this result, we can understand the large k behaviour of the Szegő projection and produce many global CR functions. As an application, we give a Szegő kernel proof of Ohsawa and Sibony's Kodaira type embedding theorem on Levi-flat CR manifolds.

We now formulate the main results. Let $(X, T^{1,0}X)$ be an orientable compact Levi-flat CR manifold of dimension $2n-1$, $n \geqslant 2$. We fix a Hermitian metric $\langle \cdot \,|\, \cdot \rangle$ on $\mathbb{C}TX$ such that $T^{1,0}X$ is orthogonal to $T^{0,1}X$. The Hermitian metric $\langle \cdot \,|\, \cdot \rangle$ on $\mathbb{C}TX$ induces a Hermitian metric $\langle \cdot \,|\, \cdot \rangle$ on the bundle $T^{*0,q}X$ of $(0,q)$ forms of X. We denote by dv_X the volume form on X induced by $\langle \cdot \,|\, \cdot \rangle$. Let (L, h) be a CR complex line bundle over X, where the Hermitian fiber metric on L is denoted by h. We will denote by R^L the curvature of L (see Definition 2.6). We say that L is positive if R^L_x is positive definite at every $x \in X$. Let $\lambda_1(x), \ldots, \lambda_{n-1}(x)$ be the eigenvalues of R^L_x with respect to $\langle \cdot \,|\, \cdot \rangle$, and set

$$\det R^L_x := \lambda_1(x) \ldots \lambda_{n-1}(x). \tag{1.1}$$

For $k > 0$, let (L^k, h^k) be the k-th tensor power of the line bundle (L, h). For $u, v \in T^{*0,q}_x X \otimes L^k_x$ we denote by $\langle u \,|\, v \rangle_{h^k}$ the induced pointwise scalar product induced by $\langle \cdot \,|\, \cdot \rangle$ and h^k. We then

get natural a global L^2 inner product $(\cdot\,|\,\cdot)_k$ on $\Omega^{0,q}(X,L^k)$,

$$(\alpha\,|\,\beta)_k := \int_X \langle\,\alpha\,|\,\beta\,\rangle_{h^k}\,dv_X\,.$$

Similarly, we have an L^2 inner product $(\cdot\,|\,\cdot)$ on $\Omega^{0,q}(X)$. We denote by $L^2_{(0,q)}(X,L^k)$ and $L^2_{(0,q)}(X)$ the completions of $\Omega^{0,q}(X,L^k)$ and $\Omega^{0,q}(X)$ with respect to $(\cdot\,|\,\cdot)_k$ and $(\cdot\,|\,\cdot)$, respectively. For $q=0$, we write $L^2(X):=L^2_{(0,0)}(X)$, $L^2(X,L^k):=L^2_{(0,0)}(X,L^k)$.

Let $\overline{\partial}_{b,k}:\mathscr{C}^\infty(X,L^k)\to\Omega^{0,1}(X,L^k)$ be the tangential Cauchy-Riemann operator cf. (2.15). We extend $\overline{\partial}_{b,k}$ to $L^2(X,L^k)$ by

$$\overline{\partial}_{b,k}:\operatorname{Dom}\overline{\partial}_{b,k}\subset L^2(X,L^k)\to L^2_{(0,1)}(X,L^k),\ \ u\longmapsto\overline{\partial}_{b,k}u\,,$$

with $\operatorname{Dom}\overline{\partial}_{b,k}:=\{u\in L^2(X,L^k);\,\overline{\partial}_{b,k}u\in L^2_{(0,1)}(X,L^k)\}$, where $\overline{\partial}_{b,k}u$ is defined in the sense of distributions. The *Szegő projection*

$$\Pi_k:L^2(X,L^k)\to\operatorname{Ker}\overline{\partial}_{b,k} \tag{1.2}$$

is the orthogonal projection with respect to $(\cdot\,|\,\cdot)_k$.

The Szegő projection Π_k is not a smoothing operator. Nevertheless, our first result shows that it enjoys the following regularity property.

Theorem 1.1. *Let X be an orientable compact Levi-flat CR manifold and let (L,h) be a positive CR line bundle on X. Then for every $\ell\in\mathbb{N}_0$ there exists $N_\ell>0$ such that for every $k\geq N_\ell$, $\Pi_k(\mathscr{C}^\infty(X,L^k))\subset\mathscr{C}^\ell(X,L^k)$ and $\Pi_k:\mathscr{C}^\infty(X,L^k)\to\mathscr{C}^\ell(X,L^k)$ is continuous.*

Let us recall now that the Szegő kernel $\Pi(x,y)$ of the boundary of a strictly pseudoconvex domain is a Fourier integral operator with complex phase, by a result of Boutet de Monvel-Sjöstrand [3] (here we consider the projection on the space of CR functions or CR sections of a fixed CR line bundle). In particular, $\Pi(x,y)$ is smooth outside the diagonal of $x=y$ and there is a precise description of the singularity on the diagonal $x=y$, where $\Pi(x,x)$ has a certain asymptotic expansion.

For a Levi-flat CR manifold we do not have such a neat characterization of the singularities of the Szegő kernel $\Pi_k(x,y)$ for fixed k. The smoothing properties of Π_k are linked to the singularities of its kernel $\Pi_k(x,y)$ and to its large k behaviour. Although it is quite difficult to describe them directly, we will show that Π_k still admits an asymptotic expansion in weak sense.

Let s be a local trivializing section of L on an open set $D\subset X$. We define the weight of the metric with respect to s to be the function $\phi\in\mathscr{C}^\infty(D)$ satisfying $|s|^2_h=e^{-2\phi}$. We have an isometry

$$U_{k,s}:L^2(D)\to L^2(D,L^k),\ \ u\longmapsto ue^{k\phi}s^k, \tag{1.3}$$

with inverse $U^{-1}_{k,s}:L^2(D,L^k)\to L^2(D)$, $\alpha\mapsto e^{-k\phi}s^{-k}\alpha$. The localization of Π_k with respect to the trivializing section s is given by

$$\Pi_{k,s}:L^2_{\rm comp}(D)\to L^2(D),\ \ \Pi_{k,s}=U^{-1}_{k,s}\Pi_kU_{k,s}, \tag{1.4}$$

where $L^2_{\rm comp}(D)$ is the subspace of elements of $L^2(D)$ with compact support in D. The second main result of this work shows that for $k\to\infty$, Π_k is rapidly decreasing outside the diagonal, and describes the singularities of Π_k in terms of an oscillatory integral.

Theorem 1.2. *Let X be an orientable compact Levi-flat CR manifold of dimension $2n-1$, $n \geq 2$. Assume that there is a positive CR line bundle L over X. Then for every $\ell \in \mathbb{N}_0$, there is $N_\ell > 0$ such that for every $k \geq N_\ell$ we have:*

(i) $\widetilde{\chi}\Pi_k\chi = O(k^{-\infty}) : \mathscr{C}^\infty(X, L^k) \to \mathscr{C}^\ell(X, L^k)$, $\forall \chi, \widetilde{\chi} \in \mathscr{C}^\infty(X)$ *with* $\operatorname{Supp}\chi \cap \operatorname{Supp}\widetilde{\chi} = \emptyset$*;*

(ii) $\Pi_{k,s} - \mathcal{S}_k = O(k^{-\infty}) : \mathscr{C}_0^\infty(D) \to \mathscr{C}^\ell(D)$*, where* $\mathcal{S}_k : \mathscr{C}_0^\infty(D) \to \mathscr{C}^\infty(D)$ *is a continuous operator and the kernel of* $\mathcal{S}_k$ *is equivalent to the oscillatory integral*

$$\mathcal{S}_k(x,y) \equiv \int e^{ik\psi(x,y,u)} s(x,y,u,k)du \mod O(k^{-\infty}), \tag{1.5}$$

where

$$\begin{aligned} & s(x,y,u,k) \sim \sum_{j=0}^{\infty} s_j(x,y,u)k^{n-j} \text{ in } S^n_{\mathrm{loc}}(1; D \times D \times \mathbb{R}), \\ & s(x,y,u,k), s_j(x,y,u) \in \mathscr{C}^\infty(D \times D \times \mathbb{R}), \ \ j = 0,1,2,\ldots, \\ & s_0(x,x,u) = \frac{1}{2}\pi^{-n}\left|\det R^L_x\right|, \quad \forall x \in D, \ \ \forall u \in \mathbb{R}, \end{aligned} \tag{1.6}$$

and the phase function $\psi \in \mathscr{C}^\infty(D \times D \times \mathbb{R})$ *satisfies* $\operatorname{Im}\psi(x,y,u) \geq 0$ *and*

$$\begin{aligned} & d_x\psi|_{(x,x,u)} = -2\mathrm{Im}\,\overline{\partial}_b\phi(x) + u\omega_0(x), \ \ x \in D, \ u \in \mathbb{R}, \\ & d_y\psi|_{(x,x,u)} = 2\mathrm{Im}\,\overline{\partial}_b\phi(x) - u\omega_0(x), \ \ x \in D, \ u \in \mathbb{R}, \\ & \frac{\partial\psi}{\partial u}(x,y,u) = 0 \text{ and } \psi(x,y,u) = 0 \text{ if and only if } x = y, \end{aligned} \tag{1.7}$$

and

$$|d_y\psi(x,y,u)| \geq c\,|u|, \ \ \forall u \in \mathbb{R}, \ \ \forall (x,y) \in D \times D, \tag{1.8}$$

where $c > 0$ is a constant. Here $\omega_0 \in \mathscr{C}^\infty(X, T^*X)$ is the positive 1-form of unit length orthogonal to $T^{*1,0}X \oplus T^{*0,1}X$, see Definition 2.4.

Note that integrating by parts with respect to y several times in (1.5) and using (1.8), we conclude that $\mathcal{S}_k$ is well-defined as a continuous operator $\mathcal{S}_k : \mathscr{C}_0^\infty(D) \to \mathscr{C}^\infty(D)$.

Using Theorem 1.2, we will show that by composing $\Pi_{k,s}$ with certain semiclassical pseudodifferential operators we obtain kernels having an asymptotic expansion in k. The freedom to choose these operators will be crucial for proving Theorem 1.4.

Let $\mathcal{A}_k$ be be a properly supported semi-classical pseudodifferential operator on D of order 0 and classical symbol (see Definition 2.3) with symbol

$$\begin{aligned} & \alpha(x,\eta,k) \sim \sum_{j=0}^{\infty} k^{-j}\alpha_j(x,\eta) \ \text{ in } S^0_{\mathrm{loc}}(1, T^*D), \\ & \alpha(x,\eta,k) = 0, \alpha_j(x,\eta) = 0, \ j = 0,1,2,\ldots, \text{for } |\eta| \geq \tfrac{1}{2}M, \text{ for some } M > 0. \end{aligned} \tag{1.9}$$

The third main result of this work is the following.

Theorem 1.3. *Let X be an orientable compact Levi-flat CR manifold of dimension $2n-1$, $n \geq 2$. Assume that there is a positive CR line bundle L over X. Then for every $\ell \in \mathbb{N}_0$, there is $N_\ell > 0$ such that for every $k \geq N_\ell$,* $(\Pi_{k,s}\mathcal{A}_k)(\cdot\,,\cdot) \in \mathscr{C}^\ell(D \times D)$ *and*

$$(\Pi_{k,s}\mathcal{A}_k)(x,y) \equiv \int e^{ik\psi(x,y,u)} a(x,y,u,k)du \mod O(k^{-\infty}) \text{ in } \mathscr{C}^\ell(D \times D), \tag{1.10}$$

where

$$
\begin{aligned}
& a(x,y,u,k) \sim \sum_{j=0}^{\infty} a_j(x,y,u)k^{n-j} \text{ in } S^n_{\rm loc}(1; D\times D\times(-M,M)),\\
(1.11) \qquad & a(x,y,u,k), a_j(x,y,u) \in \mathscr{C}^\infty_0(D\times D\times(-M,M)), \quad j=0,1,2,\ldots,\\
& a_0(x,x,u) = \frac{1}{2}\pi^{-n}\left|\det R^L_x\right| \alpha_0\big(x, u\omega_0(x) - 2{\rm Im}\,\overline{\partial}_b\phi(x)\big), \quad x\in D,\ |u|<M,
\end{aligned}
$$

and $\psi \in \mathscr{C}^\infty(D\times D\times\mathbb{R})$ *is as in Theorem 1.2.*

For more results and references about the singularities of the Szegő kernel and embedding of CR manifolds we refer to [15].

As an application of Theorem 1.1 and Theorem 1.3, we show that by projecting appropriate sections through Π_k we obtain CR sections which separate points and tangent vectors. Hence we give a Szegő kernel proof of the following result due to Ohsawa and Sibony [24, 25].

Theorem 1.4. *Let X be an orientable compact Levi-flat CR manifold of dimension $2n-1$, $n\geq 2$. Assume that there is a positive CR line bundle L over X. Then, for every $\ell\in\mathbb{N}$ there is a $M_\ell>0$ such that for every $k\geq M_\ell$, we can find N_k CR sections $s_0, s_1,\ldots,s_{N_k}\in\mathscr{C}^\ell(X,L^k)$, such that the map $X\ni x\mapsto[s_0(x),s_1(x),\ldots,s_{N_k}(x)]\in\mathbb{CP}^{N_k}$ is an embedding.*

There are no compact Levi-flat real hypersurfaces in a Stein manifold, due to the maximum principle. On the other hand, the non-existence of smooth Levi-flat hypersurfaces in complex projective spaces $\mathbb{P}^n$ attracted a lot of attention, cf. [18, 26]. The non-existence has been settled for $n\geq 3$ but a famous still open conjecture is whether this is true for $n=2$.

The paper is organized like follows. In Section 2 we collect some notations, terminology, definitions and statements we use throughout. In Section 3, we give an explicit formula for the semi-classical Kohn Laplacian $\Box^{(q)}_{b,k}$ in local coordinates and we determine the characteristic manifold for $\Box^{(q)}_{b,k}$. In Section 4 we exhibit a semi-classical Hodge decomposition for $\Box^{(q)}_{b,k}$. In Section 5, we establish some regularity for the Szegő projection and we prove Theorem 1.1. In Section 6, by using the semi-classical Hodge decomposition theorem established in Section 4 and the regularity for the Szegő projection, we prove Theorem 1.2 and Theorem 1.3. In Section 7, we prove Theorem 1.4.

2. Preliminaries

2.1. **Definitions and notations from semi-classical analysis.** We use the following notations: $\mathbb{N}=\{1,2,\ldots\}$, $\mathbb{N}_0=\mathbb{N}\cup\{0\}$, $\mathbb{R}$ is the set of real numbers, $\overline{\mathbb{R}}_+:=\{x\in\mathbb{R};\,x\geq 0\}$. For a multiindex $\alpha=(\alpha_1,\ldots,\alpha_n)\in\mathbb{N}_0^n$ we set $|\alpha|=\alpha_1+\ldots+\alpha_n$. For $x=(x_1,\ldots,x_n)$ we write

$$
\begin{aligned}
& x^\alpha = x_1^{\alpha_1}\ldots x_n^{\alpha_n},\\
& \partial_{x_j} = \frac{\partial}{\partial x_j}\,, \quad \partial^\alpha_x = \partial^{\alpha_1}_{x_1}\ldots\partial^{\alpha_n}_{x_n} = \frac{\partial^{|\alpha|}}{\partial x^\alpha}\,,\\
& D_{x_j} = \frac{1}{i}\partial_{x_j}\,, \quad D^\alpha_x = D^{\alpha_1}_{x_1}\ldots D^{\alpha_n}_{x_n}\,, \quad D_x = \frac{1}{i}\partial_x\,.
\end{aligned}
$$

Let $z = (z_1, \ldots, z_n)$, $z_j = x_{2j-1} + ix_{2j}$, $j = 1, \ldots, n$, be coordinates of $\mathbb{C}^n$. We write

$$\begin{aligned}
&z^\alpha = z_1^{\alpha_1} \ldots z_n^{\alpha_n}\,, \quad \overline{z}^\alpha = \overline{z}_1^{\alpha_1} \ldots \overline{z}_n^{\alpha_n}\,,\\
&\partial_{z_j} = \frac{\partial}{\partial z_j} = \frac{1}{2}\Big(\frac{\partial}{\partial x_{2j-1}} - i\frac{\partial}{\partial x_{2j}}\Big)\,, \quad \partial_{\overline{z}_j} = \frac{\partial}{\partial \overline{z}_j} = \frac{1}{2}\Big(\frac{\partial}{\partial x_{2j-1}} + i\frac{\partial}{\partial x_{2j}}\Big),\\
&\partial_z^\alpha = \partial_{z_1}^{\alpha_1} \ldots \partial_{z_n}^{\alpha_n} = \frac{\partial^{|\alpha|}}{\partial z^\alpha}\,, \quad \partial_{\overline{z}}^\alpha = \partial_{\overline{z}_1}^{\alpha_1} \ldots \partial_{\overline{z}_n}^{\alpha_n} = \frac{\partial^{|\alpha|}}{\partial \overline{z}^\alpha}\,.
\end{aligned}$$

Let M be a $\mathscr{C}^\infty$ orientable paracompact manifold. We let TM and T^*M denote the tangent bundle of M and the cotangent bundle of M respectively. The complexified tangent bundle of M and the complexified cotangent bundle of M will be denoted by $\mathbb{C}TM$ and $\mathbb{C}T^*M$ respectively. We write $\langle\,\cdot\,,\cdot\,\rangle$ to denote the pointwise duality between TM and T^*M. We extend $\langle\,\cdot\,,\cdot\,\rangle$ bilinearly to $\mathbb{C}TM \times \mathbb{C}T^*M$.

Let E be a $\mathscr{C}^\infty$ vector bundle over M. The fiber of E at $x \in M$ will be denoted by E_x. Let F be another vector bundle over M. We write $E \boxtimes F$ or $F \boxtimes E^*$ to denote the vector bundle over $M \times M$ with fiber over $(x, y) \in M \times M$ consisting of the linear maps from E_x to F_y.

Let $Y \subset M$ be an open set. The space of smooth sections of E over Y is denoted by $\mathscr{C}^\infty(Y, E)$ and the subspace of smooth sections with compact support is denoted by $\mathscr{C}_0^\infty(Y, E)$. Let K_M be the canonical bundle of M. The space $\mathscr{D}'(Y, E)$ of distribution sections of E is the dual of $\mathscr{C}_0^\infty(Y, E^* \otimes K_M)$. Since M is orientable we can identify $\mathscr{C}_0^\infty(Y, E^* \otimes K_M)$ to $\mathscr{C}_0^\infty(Y, E^*)$ by using a volume element on M, so we can think $\mathscr{D}'(Y, E)$ as the dual of $\mathscr{C}_0^\infty(Y, E^*)$.

Let $\mathscr{E}'(Y, E)$ be the subspace of $\mathscr{D}'(Y, E)$ whose elements have compact support in Y. For $m \in \mathbb{R}$, we let $H^m(Y, E)$ denote the Sobolev space of order m of sections of E over Y. Put

$$\begin{gathered}
H^m_{\rm loc}(Y, E) = \big\{u \in \mathscr{D}'(Y, E);\, \varphi u \in H^m(Y, E),\, \forall \varphi \in \mathscr{C}_0^\infty(Y)\big\}\,,\\
H^m_{\rm comp}(Y, E) = H^m_{\rm loc}(Y, E) \cap \mathscr{E}'(Y, E)\,.
\end{gathered}$$

We recall the Schwartz kernel theorem [9, Theorems 5.2.1, 5.2.6], [19, Thorem B.2.7]. Let E and F be smooth vector bundles over M. Let $A(\cdot, \cdot) \in \mathscr{D}'(Y \times Y, F \boxtimes E^*)$. For any fixed $u \in \mathscr{C}_0^\infty(M, E)$, the linear map $\mathscr{C}_0^\infty(M, F^*) \ni v \mapsto (A(\cdot, \cdot), v \otimes u) \in \mathbb{C}$ defines a distribution $Au \in \mathscr{D}'(Y, F)$. The operator $A : \mathscr{C}_0^\infty(M, E) \to \mathscr{D}'(M, F)$, $u \mapsto Au$, is linear and continuous.

The Schwartz kernel theorem asserts that, conversely, for any continuous linear operator $A : \mathscr{C}_0^\infty(M, E) \to \mathscr{D}'(M, F)$ there exists a unique distribution $A(\cdot, \cdot) \in \mathscr{D}'(M \times M, F \boxtimes E^*)$ such that $(Au, v) = (A(\cdot, \cdot), v \otimes u)$ for any $u \in \mathscr{C}_0^\infty(M, E)$, $v \in \mathscr{C}_0^\infty(M, F^*)$. The distribution $A(\cdot, \cdot)$ is called the Schwartz distribution kernel of A. We say that A is properly supported if the canonical projections on the two factors restricted to $\operatorname{Supp} A(\cdot, \cdot) \subset M \times M$ are proper.

The following two statements are equivalent:

(a) A can be extended to a continuous operator $A : \mathscr{E}'(M, E) \to \mathscr{C}^\infty(M, F)$,

(b) $A(\cdot, \cdot) \in \mathscr{C}^\infty(M \times M, F \boxtimes E^*)$.

If A satisfies (a) or (b), we say that A is a *smoothing operator*. Furthermore, A is smoothing if and only if for all $N \geq 0$ and $s \in \mathbb{R}$, $A : H^s_{\rm comp}(M, E) \to H^{s+N}_{\rm loc}(M, F)$ is continuous.

Let A be a smoothing operator. Then for any volume form $d\mu$, the Schwartz kernel of A is represented by a smooth kernel $K \in \mathscr{C}^\infty(M \times M, F \boxtimes E^*)$, called the Schwartz kernel of A with respect to $d\mu$, such that

$$(Au)(x) = \int_M K(x, y)u(y)\, d\mu(y)\,, \quad \text{for any } u \in \mathscr{C}_0^\infty(M, E)\,. \tag{2.1}$$

Then A can be extended as a linear continuous operator $A : \mathscr{E}'(M,E) \to \mathscr{C}^\infty(M,F)$ by setting $(Au)(x) = \big(u(\cdot), K(x,\cdot)\big)$, $x \in M$, for any $u \in \mathscr{E}'(M,E)$.

Definition 2.1. The *Szegő kernel* of the pair (X, L^k) is the the Schwartz distribution kernel $\Pi_k(\cdot,\cdot) \in \mathscr{D}'(X \times X, L^k \boxtimes L^k)$ of the Szegő projection Π_k given by (1.2).

Let W_1, W_2 be open sets in $\mathbb{R}^N$ and let E and F be complex Hermitian vector bundles over W_1 and W_2. Let $s, s' \in \mathbb{R}$ and $n_0 \in \mathbb{R}$. For a k-dependent continuous function $F_k : H^s_{\rm comp}(W_1,E) \to H^{s'}_{\rm loc}(W_2,F)$ we write

$$F_k = O(k^{n_0}) : H^s_{\rm comp}(W_1,E) \to H^{s'}_{\rm loc}(W_2,F),$$

if for any $\chi_0 \in \mathscr{C}^\infty(W_2), \chi_1 \in \mathscr{C}^\infty_0(W_1)$, there is a positive constant $c > 0$ independent of k, such that

$$\|(\chi_0 F_k \chi_1)u\|_{s'} \leq c k^{n_0} \|u\|_s\,, \quad \forall u \in H^s_{\rm loc}(W_1,E), \tag{2.2}$$

where $\|\cdot\|_s$ denotes the usual Sobolev norm of order s. We write

$$F_k = O(k^{-\infty}) : H^s_{\rm comp}(W_1,E) \to H^{s'}_{\rm loc}(W_2,F),$$

if $F_k = O(k^{-N}) : H^s_{\rm comp}(W_1,E) \to H^{s'}_{\rm loc}(W_2,F)$, for every $N > 0$. Similarly, let $\ell \in \mathbb{N}$, for a k-dependent continuous function $G_k : \mathscr{C}^\infty_0(W_1,E) \to \mathscr{C}^\ell(W_2,F)$ we write

$$G_k = O(k^{-\infty}) : \mathscr{C}^\infty_0(W_1,E) \to \mathscr{C}^\ell(W_2,F),$$

if for any $\chi_0 \in \mathscr{C}^\infty(W_2), \chi_1 \in \mathscr{C}^\infty_0(W_1)$ and $N > 0$, there are positive constants $c > 0$ and $M \in \mathbb{N}_0$ independent of k, such that

$$\|(\chi_0 G_k \chi_1)u\|_{\mathscr{C}^\ell(W_2,F)} \leq c k^{-N} \|u\|_{\mathscr{C}^M(W_1,E)}\,, \quad \forall u \in \mathscr{C}^\infty_0(W_1,E), \tag{2.3}$$

A k-dependent continuous operator $A_k : \mathscr{C}^\infty_0(W_1,E) \to \mathscr{D}'(W_2,F)$ is called k-negligible on $W_2 \times W_1$ if for k large enough A_k is smoothing and for any $K \Subset W_2 \times W_1$, any multi-indices α, β and any $N \in \mathbb{N}$ there exists $C_{K,\alpha,\beta,N} > 0$ such that

$$\left|\partial^\alpha_x \partial^\beta_y A_k(x,y)\right| \leq C_{K,\alpha,\beta,N} k^{-N}\,, \text{ on } K. \tag{2.4}$$

Let $C_k : \mathscr{C}^\infty_0(W_1,E) \to \mathscr{D}'(W_2,F)$ be another k-dependent continuous operator. We write $A_k \equiv C_k \mod O(k^{-\infty})$ (on $W_2 \times W_1$) or $A_k(x,y) \equiv C_k(x,y) \mod O(k^{-\infty})$ (on $W_2 \times W_1$) if $A_k - C_k$ is k-negligible on $W_2 \times W_1$.

Similarly, for $\ell \in \mathbb{N}_0$, $A_k : \mathscr{C}^\infty_0(W_1,E) \to \mathscr{D}'(W_2,F)$ is called k-negligible in the $\mathscr{C}^\ell$ norm on $W_2 \times W_1$ if $A_k(x,y) \in \mathscr{C}^\ell(W_2 \times W_1, E_y \boxtimes F_x)$ for k large and (2.4) holds for multi-indices α, β with $|\alpha| + |\beta| \leq \ell$.

Let $C_k : \mathscr{C}^\infty_0(W_1,E) \to \mathscr{D}'(W_2,F)$ be another k-dependent continuous operator. We write $A_k \equiv C_k \mod O(k^{-\infty})$ in the $\mathscr{C}^\ell$ norm (on $W_2 \times W_1$) or $A_k(x,y) \equiv C_k(x,y) \mod O(k^{-\infty})$ in $\mathscr{C}^\ell$ norm (on $W_2 \times W_1$) if $A_k - C_k$ is k-negligible in $\mathscr{C}^\ell$ norm on $W_2 \times W_1$.

Let $B_k : L^2(X, L^k) \to L^2(X, L^k)$ be a continuous operator. Let s, s_1 be local trivializing sections of L on open sets $D_0 \Subset M$, $D_1 \Subset M$ respectively, $|s|^2_h = e^{-2\phi}$, $|s_1|^2_h = e^{-2\phi_1}$. The localized operator (with respect to the trivializing sections s and s_1) of B_k is given by

$$B_{k,s,s_1} : L^2(D_1) \cap \mathscr{E}'(D_1) \to L^2(D),\ \ u \longmapsto e^{-k\phi} s^{-k} B_k(s_1^k e^{k\phi_1} u) = U^{-1}_{k,s} B_k U_{k,s_1}, \tag{2.5}$$

and let $B_{k,s,s_1}(x,y) \in \mathscr{D}'(D \times D_1)$ be the distribution kernel of B_{k,s,s_1}. We write

$$B_k = O(k^{n_0}) : H^s(X, L^k) \to H^{s'}(X, L^k),\ \ n_0 \in \mathbb{R},$$

if for all local trivializing sections s, s_1 on D and D_1 respectively, we have

$$B_{k,s,s_1} = O(k^{n_0}) : H^s_{\rm comp}(D_1) \to H^{s'}_{\rm loc}(D).$$

We write

$$B_k = O(k^{-\infty}) : H^s(X, L^k) \to H^{s'}(X, L^k), \ \ n_0 \in \mathbb{R},$$

if for all local trivializing sections s, s_1 on D and D_1 respectively, we have

$$B_{k,s,s_1} = O(k^{-\infty}) : H^s_{\rm comp}(D_1) \to H^{s'}_{\rm loc}(D).$$

Fix $\ell \in \mathbb{N}$. We write

$$B_k = O(k^{-\infty}) : \mathscr{C}^\infty(X, L^k) \to \mathscr{C}^\ell(X, L^k),$$

if for all local trivializing sections s, s_1 on D and D_1 respectively, we have

$$B_{k,s,s_1} = O(k^{-\infty}) : \mathscr{C}^\infty_0(D_1) \to \mathscr{C}^\ell(D).$$

We recall semi-classical symbol spaces (see Dimassi-Sjöstrand [7, Chapter 8]):

Definition 2.2. Let W be an open set in $\mathbb{R}^N$. Let

$$S(1; W) := \Big\{ a \in \mathscr{C}^\infty(W) \,|\, \forall \alpha \in \mathbb{N}_0^N : \sup_{x \in W} |\partial^\alpha a(x)| < \infty \Big\},$$

$$S^0_{\rm loc}(1; W) := \Big\{ (a(\cdot, k))_{k \in \mathbb{N}} \,|\, \forall \alpha \in \mathbb{N}_0^N, \forall \chi \in \mathscr{C}^\infty_0(W) \ : \ \sup_{k \in \mathbb{N}} \sup_{x \in W} |\partial^\alpha a(x, k)| < \infty \Big\} .$$

For $m \in \mathbb{R}$ let

$$S^m_{\rm loc}(1; W) = \Big\{ (a(\cdot, k))_{k \in \mathbb{N}} \,|\, (k^{-m} a(\cdot, k)) \in S^0_{\rm loc}(1; W) \Big\} .$$

Hence $a(\cdot, k)) \in S^m_{\rm loc}(1; W)$ if for every $\alpha \in \mathbb{N}_0^N$ and $\chi \in \mathscr{C}^\infty_0(W)$, there exists $C_\alpha > 0$, such that $|\partial^\alpha(\chi a(\cdot, k))| \le C_\alpha k^m$ on W.

Consider a sequence $a_j \in S^{m_j}_{\rm loc}(1)$, $j \in \mathbb{N}_0$, where $m_j \searrow -\infty$, and let $a \in S^{m_0}_{\rm loc}(1)$. We say that

$$a(\cdot, k) \sim \sum_{j=0}^\infty a_j(\cdot, k), \ \text{ in } S^{m_0}_{\rm loc}(1),$$

if for every $\ell \in \mathbb{N}_0$ we have $a - \sum_{j=0}^\ell a_j \in S^{m_{\ell+1}}_{\rm loc}(1)$. For a given sequence a_j as above, we can always find such an asymptotic sum a, which is unique up to an element in $S^{-\infty}_{\rm loc}(1) = S^{-\infty}_{\rm loc}(1; W) := \cap_m S^m_{\rm loc}(1)$.

We say that $a(\cdot, k) \in S^m_{\rm loc}(1)$ is a classical symbol on W of order m if

$$a(\cdot, k) \sim \sum_{j=0}^\infty k^{m-j} a_j \text{ in } S^{m_0}_{\rm loc}(1), \ \ a_j(x) \in S_{\rm loc}(1), \ j = 0, 1 \ldots . \tag{2.6}$$

The set of all classical symbols on W of order m_0 is denoted by $S^{m_0}_{\rm loc\,,cl}(1) = S^{m_0}_{\rm loc\,,cl}(1; W)$.

Definition 2.3. Let W be an open set in $\mathbb{R}^N$. A semi-classical pseudodifferential operator on W of order m and classical symbol is a k-dependent continuous operator $A_k : \mathscr{C}^\infty_0(W) \to \mathscr{C}^\infty(W)$ such that the distribution kernel $A_k(x, y)$ is given by the oscillatory integral

$$\begin{aligned} A_k(x, y) \equiv & \frac{k^N}{(2\pi)^N} \int e^{ik\langle x-y, \eta \rangle} a(x, y, \eta, k) d\eta \mod O(k^{-\infty}), \\ & a(x, y, \eta, k) \in S^m_{\rm loc\,,cl}(1; W \times W \times \mathbb{R}^N). \end{aligned} \tag{2.7}$$

We shall identify A_k with $A_k(x, y)$. It is clear that A_k has a unique continuous extension $A_k : \mathscr{E}'(W) \to \mathscr{D}'(W)$. For $u \in \mathscr{C}_0^\infty(W)$ we have

$$A_k u(x) \equiv \frac{k^N}{(2\pi)^N} \int e^{ik\langle x-y,\eta\rangle} \alpha(x, \eta, k) u(y) d\eta \mod O(k^{-\infty}), \tag{2.8}$$

with symbol

$$\alpha(x, \eta, k) \in S^m_{\mathrm{loc,cl}}(1; W \times \mathbb{R}^N) = S^m_{\mathrm{loc,cl}}(1; T^*W). \tag{2.9}$$

2.2. **CR manifolds and bundles.** A Cauchy-Riemann (CR) manifold (of hypersurface type) is a pair $(X, T^{1,0}X)$ where X is a smooth manifold of dimension $2n-1$, $n \geqslant 2$, and $T^{1,0}X$ is a sub-bundle of the complexified tangent bundle $\mathbb{C}TX := \mathbb{C} \otimes TX$, of rank $(n-1)$, such that $T^{1,0}X \cap \overline{T^{1,0}X} = \{0\}$ and the set of smooth sections of $T^{1,0}X$ is closed under the Lie bracket. We call $T^{1,0}X$ the CR structure of X and we denote $T^{0,1}X := \overline{T^{1,0}X}$.

We say that $(X, T^{1,0}X)$ is a *Levi-flat CR manifold* if the set of smooth sections of $T^{1,0}X \oplus T^{0,1}X$ is closed under the Lie bracket. If X is Levi-flat, there exists a smooth foliation of X, of real codimension one and whose leaves are complex manifolds: it is obtained by integrating the distribution $(T^{1,0}X \oplus T^{0,1}X) \cap TX$.

In this paper, we assume throughout that X is an orientable Levi-flat manifold.

Fix a smooth Hermitian metric $\langle \cdot | \cdot \rangle$ on $\mathbb{C}TX$ so that $T^{1,0}X$ is orthogonal to $T^{0,1}X$ and $\langle u | v \rangle$ is real if u, v are real tangent vectors. Then locally there is a real non-vanishing vector field T of length one which is pointwise orthogonal to $T^{1,0}X \oplus T^{0,1}X$. T is unique up to the choice of sign. For $u \in \mathbb{C}TX$, we write $|u|^2 := \langle u | u \rangle$. Denote by $T^{*1,0}X$ and $T^{*0,1}X$ the dual bundles of $T^{1,0}X$ and $T^{0,1}X$, respectively. They can be identified with subbundles of the complexified cotangent bundle $\mathbb{C}T^*X$.

Define the vector bundle of $(0, q)$-forms by $T^{*0,q}X := \Lambda^q T^{*0,1}X$. The Hermitian metric $\langle \cdot | \cdot \rangle$ on $\mathbb{C}TX$ induces, by duality, a Hermitian metric on $\mathbb{C}T^*X$ and also on the bundles of $(0, q)$ forms $T^{*0,q}X$, $q = 0, 1, \ldots, n-1$. We shall also denote all these induced metrics by $\langle \cdot | \cdot \rangle$. Let $\Omega^{0,q}(D)$ denote the space of smooth sections of $T^{*0,q}X$ over D and let $\Omega_0^{0,q}(D)$ be the subspace of $\Omega^{0,q}(D)$ whose elements have compact support in D. Similarly, if E is a vector bundle over D, then we let $\Omega^{0,q}(D, E)$ denote the space of smooth sections of $T^{*0,q}X \otimes E$ over D and let $\Omega_0^{0,q}(D, E)$ be the subspace of $\Omega^{0,q}(D, E)$ whose elements have compact support in D.

Locally we can choose an orthonormal frame $\omega_1, \ldots, \omega_{n-1}$ of the bundle $T^{*1,0}X$. Then $\overline{\omega}_1, \ldots, \overline{\omega}_{n-1}$ is an orthonormal frame of the bundle $T^{*0,1}X$. The real $(2n-2)$-form $\omega = i^{n-1}\omega_1 \wedge \overline{\omega}_1 \wedge \cdots \wedge \omega_{n-1} \wedge \overline{\omega}_{n-1}$ is independent of the choice of the orthonormal frame. Thus ω is globally defined. Locally there is a real 1-form ω_0 of length one which is orthogonal to $T^{*1,0}X \oplus T^{*0,1}X$. The form ω_0 is unique up to the choice of sign. Since X is orientable, there is a nowhere vanishing $(2n-1)$ form Q on X. Thus, ω_0 can be specified uniquely by requiring that $\omega \wedge \omega_0 = fQ$, where f is a positive function. Therefore ω_0, so chosen, is globally defined.

Definition 2.4. We call ω_0 the positive 1-form of unit length orthogonal to $T^{*1,0}X \oplus T^{*0,1}X$.

We choose a vector field T so that

$$|T| = 1\,, \quad \langle T\,,\,\omega_0 \rangle = -1\,. \tag{2.10}$$

Therefore T is uniquely determined. We call T the uniquely determined global real vector field. We have the pointwise orthogonal decompositions:

$$\begin{aligned} \mathbb{C}T^*X &= T^{*1,0}X \oplus T^{*0,1}X \oplus \{\lambda\omega_0;\, \lambda \in \mathbb{C}\}\,, \\ \mathbb{C}TX &= T^{1,0}X \oplus T^{0,1}X \oplus \{\lambda T;\, \lambda \in \mathbb{C}\}\,. \end{aligned} \tag{2.11}$$

Let

$$\overline{\partial}_b : \Omega^{0,q}(X) \to \Omega^{0,q+1}(X) \tag{2.12}$$

be the tangential Cauchy-Riemann operator. Let $U \subset X$ be an open set. We say that a function $u \in \mathscr{C}^\infty(U)$ is Cauchy-Riemann (CR for short) (on U) if $\overline{\partial}_b u = 0$.

Definition 2.5. Let L be a complex line bundle over a CR manifold X. We say that L is a Cauchy-Riemann (CR for short) (complex) line bundle over X if its transition functions are CR.

If X is Levi-flat, then the restriction a CR line bundle to any leaf Y of the Levi-foliation is a holomorphic line bundle.

From now on, we let (L, h) be a CR line bundle over X, where the Hermitian fiber metric on L is denoted by h. We will denote by ϕ the local weights of the Hermitian metric. More precisely, if s is a local trivializing section of L on an open subset $D \subset X$, then the local weight of h with respect to s is the function $\phi \in \mathscr{C}^\infty(D, \mathbb{R})$ for which

$$|s(x)|_h^2 = e^{-2\phi(x)}\,, \quad x \in D. \tag{2.13}$$

Definition 2.6. Let s be a local trivializing section of L on an open subset $D \subset X$ and ϕ the corresponding local weight as in (2.13). For $p \in D$, we define the Hermitian quadratic form M_p^ϕ on $T_p^{1,0}X$ by

$$M_p^\phi(U, V) = \Big\langle U \wedge \overline{V}, d\big(\overline{\partial}_b\phi - \partial_b\phi\big)(p)\Big\rangle,\ \ U, V \in T_p^{1,0}X, \tag{2.14}$$

where d is the usual exterior derivative and $\overline{\partial_b\phi} = \overline{\partial}_b\overline{\phi}$. Since X is Levi-flat, the definition of M_p^ϕ does not depend on the choice of local trivializations (see [14, Proposition 4.2]). Hence there exists a smooth section R^L of the bundle of Hermitian forms on $T^{1,0}X$ such that $R^L|_D = M^\phi$. We call R^L the curvature of (L, h). We say that (L, h), or R^L, is positive if R^L_x is positive definite, for every $x \in X$. We say that L is a positive CR line bundle over X if there is a Hermitian fiber metric h on L such that the induced curvature R^L is positive.

In this paper, we assume that L is a positive CR line bundle over a Levi-flat CR manifold X and we fix a Hermitian fiber metric h of L such that the induced curvature R^L is positive. Note that a positive line bundle (L, h) in the sense of Definition 2.6 is positive along the leaves of the Levi-foliation: its restriction $(L, h)|_Y$ to any leaf Y is positive (that is, the curvature of the associated Chern connection is positive).

Let L^k, $k > 0$, be the k-th tensor power of the line bundle L. The Hermitian fiber metric on L induces a Hermitian fiber metric on L^k that we shall denote by h^k. If s is a local trivializing section of L then s^k is a local trivializing section of L^k. We write $\overline{\partial}_{b,k}$ to denote the tangential Cauchy-Riemann operator acting on forms with values in L^k, defined locally by

$$\overline{\partial}_{b,k} : \Omega^{0,q}(X, L^k) \to \Omega^{0,q+1}(X, L^k)\,, \quad \overline{\partial}_{b,k}(s^k u) := s^k\overline{\partial}_b u, \tag{2.15}$$

where s is a local trivialization of L on an open subset $D \subset X$ and $u \in \Omega^{0,q}(D)$.

3. The semi-classical Kohn Laplacian

We first introduce some notations. For $v \in T^{*0,q}X$ we denote by $v\wedge : T^{*0,\bullet}X \to T^{*0,\bullet+q}X$ the exterion multiplication by v and let $v^{\wedge,*} : T^{*0,\bullet}X \to T^{*0,\bullet-q}X$ be the adjoint of $v\wedge$ with respect to $\langle\,\cdot\,|\,\cdot\,\rangle$. Hence, $\langle\, v \wedge u \,|\, g \,\rangle = \langle\, u \,|\, v^{\wedge,*}g \,\rangle$, $\forall u \in T^{*0,p}X$, $g \in T^{*0,p+q}X$.

For any $r = 0, 1, \ldots, n-2$, we write

$$\overline{\partial}^*_{b,k} : \mathrm{Dom}\, \overline{\partial}^*_{b,k} \subset L^2_{(0,r+1)}(X, L^k) \to L^2_{(0,r)}(X, L^k) \tag{3.1}$$

to denote the Hilbert space adjoint of $\overline{\partial}_{b,k}$ in the L^2 space with respect to $(\,\cdot\,|\,\cdot\,)_k$. Let $\Box^{(q)}_{b,k}$ denote the (Gaffney extension of the) *Kohn Laplacian* given by

(3.2)
$$\mathrm{Dom}\, \Box^{(q)}_{b,k} = \left\{ s \in \mathrm{Dom}\, \overline{\partial}_{b,k} \cap \mathrm{Dom}\, \overline{\partial}^*_{b,k} \subset L^2_{(0,q)}(X, L^k);\ \overline{\partial}_{b,k} s \in \mathrm{Dom}\, \overline{\partial}^*_{b,k},\ \overline{\partial}^*_{b,k} s \in \mathrm{Dom}\, \overline{\partial}_{b,k} \right\},$$

and $\Box^{(q)}_{b,k} s = \overline{\partial}_{b,k}\overline{\partial}^*_{b,k} s + \overline{\partial}^*_{b,k}\overline{\partial}_{b,k} s$ for $s \in \mathrm{Dom}\, \Box^{(q)}_{b,k}$. Note that $\mathrm{Ker}\, \Box^{(0)}_{b,k} = \mathrm{Ker}\, \overline{\partial}_{b,k}$. By a result of Gaffney [19, Proposition 3.1.2], $\Box^{(q)}_{b,k}$ is a positive self-adjoint operator.

Let s be a local trivializing of L on an open subset $D \subset X$. By using the map (1.3) we have the unitary identifications:

$$\begin{cases} \mathscr{C}^\infty_0(D, T^{*0,q}X) \longleftrightarrow \mathscr{C}^\infty_0(D, L^k \otimes T^{*0,q}X) \\ \qquad u \longleftrightarrow \widetilde{u} = U_{k,s} u, \quad u = U^{-1}_{k,s}\widetilde{u}, \\ \qquad \overline{\partial}_{s,k} \longleftrightarrow \overline{\partial}_{b,k}, \quad \overline{\partial}_{s,k} u = U^{-1}_{k,s}\overline{\partial}_{b,k} U_{k,s}, \\ \qquad \overline{\partial}^*_{s,k} \longleftrightarrow \overline{\partial}^*_{b,k}, \quad \overline{\partial}^*_{s,k} u = U^{-1}_{k,s}\overline{\partial}^*_{b,k} U_{k,s}, \\ \qquad \Box^{(q)}_{s,k} \longleftrightarrow \Box^{(q)}_{b,k}, \quad \Box^{(q)}_{s,k} u = U^{-1}_{k,s}\Box^{(q)}_{b,k} U_{k,s}. \end{cases} \tag{3.3}$$

It is easy to see that

$$\overline{\partial}_{s,k} = \overline{\partial}_b + k(\overline{\partial}_b\phi)\wedge, \quad \overline{\partial}^*_{s,k} = \overline{\partial}^*_b + k(\overline{\partial}_b\phi)^{\wedge,*} \tag{3.4}$$

and

$$\Box^{(q)}_{s,k} = \overline{\partial}_{s,k}\overline{\partial}^*_{s,k} + \overline{\partial}^*_{s,k}\overline{\partial}_{s,k}. \tag{3.5}$$

Here $\overline{\partial}^*_b : \mathscr{C}^\infty(X, T^{*0,q+1}X) \to \mathscr{C}^\infty(X, T^{*0,q}X)$ is the formal adjoint of $\overline{\partial}_b$ with respect to $(\,\cdot\,|\,\cdot\,)$.

Let us choose a smooth orthonormal frame $\{e_j\}_{j=1}^{n-1}$ for $T^{*0,1}X$ on D. Let $\{Z_j\}_{j=1}^{n-1}$ denote the dual frame of $T^{0,1}X$. Let Z^*_j be the formal adjoint of Z_j with respect to $(\,\cdot\,|\,\cdot\,)$, $j = 1, \ldots, n-1$, that is, $(Z_j f \mid h) = (f \mid Z^*_j h)$, $f, h \in \mathscr{C}^\infty_0(D)$.

Proposition 3.1 ([13, Proposition 3.1])**.** *With the notations used before, using the identification* (3.3)*, we can identify the Kohn Laplacian* $\Box^{(q)}_{b,k}$ *with*

$$\begin{aligned} \Box^{(q)}_{s,k} &= \overline{\partial}_{s,k}\overline{\partial}^*_{s,k} + \overline{\partial}^*_{s,k}\overline{\partial}_{s,k} \\ &= \sum_{j=1}^{n-1} (Z^*_j + k\overline{Z}_j(\phi))(Z_j + kZ_j(\phi)) \\ &\quad + \sum_{j,t=1}^{n-1} e_j \wedge e_t^{\wedge,*} \circ [Z_j + kZ_j(\phi), Z^*_t + k\overline{Z}_t(\phi)] \\ &\quad + \varepsilon(Z + kZ(\phi)) + \varepsilon(Z^* + k\overline{Z}(\phi)) + f, \end{aligned} \tag{3.6}$$

where $\varepsilon(Z + kZ(\phi))$ *denotes remainder terms of the form* $\sum a_j(Z_j + kZ_j(\phi))$ *with* a_j *smooth, matrix-valued and independent of* k*, for all* j*, and similarly for* $\varepsilon(Z^* + k\overline{Z}(\phi))$ *and* f *is a smooth function independent of* k*.*

The operator $\Box^{(q)}_{s,k}$ will be called the *localized Kohn Laplacian*. Until further notice, we work with some real local coordinates $x = (x_1, \ldots, x_{2n-1})$ defined on D. Let $\xi = (\xi_1, \ldots, \xi_{2n-1})$ denote the dual variables of x. Then (x, ξ) are local coordinates of the cotangent bundle T^*D. Let $q_j(x, \xi)$ be the semi-classical principal symbol of $Z_j + kZ_j(\phi)$, $j = 1, \ldots, n-1$. If $r_j(x, \xi)$ denotes the principal symbol of Z_j, then $q_j(x, \xi) = r_j(x, \xi) + Z_j(\phi)$. The semi-classical principal symbol of $\Box^{(q)}_{s,k}$ is given by

$$p_0 = \sum_{j=1}^{n-1} \overline{q}_j q_j. \tag{3.7}$$

The characteristic manifold Σ of $\Box^{(q)}_{s,k}$ is

$$\begin{aligned}\Sigma &= \{(x, \xi) \in T^*D;\, p_0(x, \xi) = 0\} \\ &= \left\{(x, \xi) \in T^*D;\, q_1(x, \xi) = \cdots = q_{n-1}(x, \xi) = \overline{q}_1(x, \xi) = \cdots = \overline{q}_{n-1}(x, \xi) = 0\right\}.\end{aligned} \tag{3.8}$$

From (3.8), we see that p_0 vanishes to second order at Σ. The following is also well-known [13, Proposition 3.2]

Proposition 3.2. *We have*

$$\Sigma = \left\{(x, \xi) \in T^*D;\, \xi = \lambda\omega_0(x) - 2\mathrm{Im}\,\overline{\partial}_b\phi(x), \lambda \in \mathbb{R}\right\}. \tag{3.9}$$

Let $\sigma = d\xi \wedge dx$ denote the canonical two form on T^*D. We are interested in whether σ is non-degenerate at $\rho \in \Sigma$. We recall that σ is non-degenerate at $\rho \in \Sigma$ if $\sigma(u, v) = 0$ for all $v \in \mathbb{C}T_\rho\Sigma$, where $u \in \mathbb{C}T_\rho\Sigma$, then $u = 0$. We recall that we work with the assumption that X is Levi-flat. From this observation and Theorem 3.5 in [13] , we conclude that:

Theorem 3.3. *σ is non-degenerate at every point of Σ.*

4. Semi-classical Hodge decomposition for the localized Kohn Laplacian

In this section, we will apply the method introduced in [13] to establish semi-classical Hodge decomposition theorems for $\Box^{(0)}_{s,k}$. Since the procedure is similar, we will only give the outline. We refer the reader to [13, Section 4 and Section 5] for the details.

4.1. The heat equation for the local operator $\Box^{(0)}_s$.

We first introduce some notations. Let Ω be an open set in $\mathbb{R}^N$ and let f, g be positive continuous functions on Ω. We write $f \asymp g$ if for every compact set $K \subset \Omega$ there is a constant $c_K > 0$ such that $f \leq c_K g$ and $g \leq c_K f$ on K.

Let s be a local trivializing section of L on an open subset $D \Subset X$ and $|s|^2_h = e^{-2\phi}$. In this section, we work with some real local coordinates $x = (x_1, \ldots, x_{2n-1})$ defined on D. We write $\xi = (\xi_1, \ldots, \xi_{2n-1})$ or $\eta = (\eta_1, \ldots, \eta_{2n-1})$ to denote the dual coordinates of x. We consider the domain $\widehat{D} := D \times \mathbb{R}$. We write $\widehat{x} := (x, x_{2n}) = (x_1, x_2, \ldots, x_{2n-1}, x_{2n})$ to denote the coordinates of $D \times \mathbb{R}$, where x_{2n} is the coordinate of $\mathbb{R}$. We write $\widehat{\xi} := (\xi, \xi_{2n})$ or $\widehat{\eta} := (\eta, \eta_{2n})$ to denote the dual coordinates of $\widehat{x}$, where ξ_{2n} and η_{2n} denote the dual coordinate of x_{2n}. We shall use the following notations: $\langle x, \eta\rangle := \sum_{j=1}^{2n-1} x_j\eta_j$, $\langle x, \xi\rangle := \sum_{j=1}^{2n-1} x_j\xi_j$, $\langle \widehat{x}, \widehat{\eta}\rangle := \sum_{j=1}^{2n} x_j\eta_j$, $\langle \widehat{x}, \widehat{\xi}\rangle := \sum_{j=1}^{2n} x_j\xi_j$.

Let $T^{*0,q}\widehat{D}$ be the bundle with fiber $T^{*0,q}_{\widehat{x}}\widehat{D} := \{u \in T^{*0,q}_x D, \widehat{x} = (x, x_{2n})\}$ at $\widehat{x} \in \widehat{D}$. From now on, for every point $\widehat{x} = (x, x_{2n}) \in \widehat{D}$, we identify $T^{*0,q}_{\widehat{x}}\widehat{D}$ with $T^{*0,q}_x X$. Let $\langle \cdot | \cdot \rangle$ be the

Hermitian metric on $\mathbb{C}T^*\widehat{D}$ given by $\langle\,\widehat{\xi}\,|\,\widehat{\eta}\,\rangle = \langle\,\xi\,|\,\eta\,\rangle + \xi_{2n}\overline{\eta_{2n}}$, $(\widehat{x},\widehat{\xi}),(\widehat{x},\widehat{\eta}) \in \mathbb{C}T^*\widehat{D}$. Let $\Omega^{0,q}(\widehat{D})$ denote the space of smooth sections of $T^{*0,q}\widehat{D}$ over $\widehat{D}$ and put

$$\Omega_0^{0,q}(\widehat{D}) := \Omega^{0,q}(\widehat{D}) \cap \mathscr{E}'(\widehat{D}, T^{*0,q}\widehat{D}).$$

Using the identification

$$ku(x) = e^{-ikx_{2n}}\left(-i\frac{\partial}{\partial x_{2n}}\left(e^{ikx_{2n}}u\right)(x)\right),\ \ u \in \Omega^{0,q}(D),$$

we consider the following operators

$$\begin{aligned}&\overline{\partial}_s : \Omega^{0,r}(\widehat{D}) \to \Omega^{0,r+1}(\widehat{D}),\ \ \overline{\partial}_{s,k}u = e^{-ikx_{2n+1}}\overline{\partial}_s(ue^{ikx_{2n}}),\ \ u \in \Omega^{0,r}(D),\\ &\overline{\partial}^*_s : \Omega^{0,r+1}(\widehat{D}) \to \Omega^{0,r}(\widehat{D}),\ \ \overline{\partial}^*_{s,k}u = e^{-ikx_{2n+1}}\overline{\partial}^*_s(ue^{ikx_{2n}}),\ \ u \in \Omega^{0,r+1}(D),\end{aligned} \tag{4.1}$$

where $r = 0,1,\ldots,n-1$ and $\overline{\partial}_{s,k}$, $\overline{\partial}^*_{s,k}$ are given by (3.3). From (3.4) it is easy to see that

$$\begin{aligned}\overline{\partial}_s &= \sum_{j=1}^{n-1}\left(e_j\wedge\left(Z_j - iZ_j(\phi)\frac{\partial}{\partial x_{2n}}\right) + (\overline{\partial}_b e_j)\wedge e_j^{\wedge,*}\right),\\ \overline{\partial}^*_s &= \sum_{j=1}^{n-1}\left(e_j^{\wedge,*}\left(Z^*_j - i\overline{Z}_j(\phi)\frac{\partial}{\partial x_{2n}}\right) + e_j\wedge(\overline{\partial}_b e_j)^{\wedge,*}\right),\end{aligned} \tag{4.2}$$

where $Z_1,\ldots,Z_{n-1}$, $Z^*_1,\ldots,Z^*_{n-1}$ and $e_1,\ldots,e_{n-1}$ are as in Proposition 3.1. Put

$$\Box^{(q)}_s := \overline{\partial}_s\overline{\partial}^*_s + \overline{\partial}^*_s\overline{\partial}_s : \Omega^{0,q}(\widehat{D}) \to \Omega^{0,q}(\widehat{D}). \tag{4.3}$$

From (4.1), we have

$$\Box^{(q)}_{s,k}u = e^{-ikx_{2n}}\Box^{(q)}_s(ue^{ikx_{2n}}),\ \ \forall u \in \Omega^{0,q}(D), \tag{4.4}$$

where $\Box^{(q)}_{s,k}$ is given by (3.3). Let $u \in \Omega^{0,q}_0(\widehat{D})$. Note that

$$k\int e^{-ikx_{2n}}u(x)dx_{2n} = \int i\frac{\partial}{\partial x_{2n}}(e^{-ikx_{2n}})u(x)dx_{2n} = \int e^{-ikx_{2n}}\Big(-i\frac{\partial u}{\partial x_{2n}}(x)\Big)dx_{2n}.$$

From this observation and the explicit formulas for $\overline{\partial}_{s,k}$, $\overline{\partial}^*_{s,k}$, $\overline{\partial}_s$ and $\overline{\partial}^*_s$ (see (3.4) and (4.2)), we conclude that

$$\Box^{(q)}_{s,k}\int e^{-ikx_{2n}}u(x)dx_{2n} = \int e^{-ikx_{2n}}(\Box^{(q)}_s u)(x)dx_{2n},\ \ u \in \Omega^{0,q}_0(\widehat{D}). \tag{4.5}$$

As in Proposition 4.1 in [13], we have:

Proposition 4.1. *With the notations used before, we have*

$$\begin{aligned}\Box^{(q)}_s &= \overline{\partial}_s\overline{\partial}^*_s + \overline{\partial}^*_s\overline{\partial}_s\\ &= \sum_{j=1}^{n-1}\left(Z^*_j - i\overline{Z}_j(\phi)\frac{\partial}{\partial x_{2n}}\right)\left(Z_j - iZ_j(\phi)\frac{\partial}{\partial x_{2n}}\right)\\ &+ \sum_{j,t=1}^{n-1} e_j\wedge e_t^{\wedge,*}\left[Z_j - iZ_j(\phi)\frac{\partial}{\partial x_{2n}}, Z^*_t - i\overline{Z}_t(\phi)\frac{\partial}{\partial x_{2n}}\right]\\ &+ \varepsilon\left(Z - iZ(\phi)\frac{\partial}{\partial x_{2n}}\right) + \varepsilon\left(Z^* - i\overline{Z}(\phi)\frac{\partial}{\partial x_{2n}}\right) + \textit{zero order terms},\end{aligned} \tag{4.6}$$

where $\varepsilon(Z - iZ(\phi)\frac{\partial}{\partial x_{2n}})$ denotes remainder terms of the form $\sum a_j(Z_j - iZ_j(\phi)\frac{\partial}{\partial x_{2n}})$ with a_j smooth, matrix-valued, for all j, and similarly for $\varepsilon(Z^* - i\overline{Z}(\phi)\frac{\partial}{\partial x_{2n}})$.

In this paper, we will only consider $q = 0$. Consider the problem

$$
\begin{cases} (\partial_t + \Box_s^{(0)})u(t, \widehat{x}) = 0 \quad \text{in } \mathbb{R}_+ \times \widehat{D}, \\ u(0, \widehat{x}) = v(\widehat{x}). \end{cases} \tag{4.7}
$$

Definition 4.2. We say that $a(t, \widehat{x}, \widehat{\eta}) \in \mathscr{C}^\infty(\overline{\mathbb{R}}_+ \times T^*\widehat{D})$ is quasi-homogeneous of degree j if $a(t, \widehat{x}, \lambda\widehat{\eta}) = \lambda^j a(\lambda t, \widehat{x}, \widehat{\eta})$ for all $\lambda > 0$, $|\widehat{\eta}| \geq 1$. We say that $b(\widehat{x}, \widehat{\eta}) \in \mathscr{C}^\infty(T^*\widehat{D})$ is positively homogeneous of degree j if $b(\widehat{x}, \lambda\widehat{\eta}) = \lambda^j b(\widehat{x}, \widehat{\eta})$ for all $\lambda > 0$, $|\widehat{\eta}| \geq 1$.

We look for an approximate solution of (4.7) of the form $u(t, \widehat{x}) = A(t)v(\widehat{x})$,

$$
A(t)v(\widehat{x}) = \frac{1}{(2\pi)^{2n}} \iint e^{i(\Psi(t,\widehat{x},\widehat{\eta}) - \langle \widehat{y}, \widehat{\eta}\rangle)} a(t, \widehat{x}, \widehat{\eta}) v(\widehat{y}) d\widehat{y} d\widehat{\eta} \tag{4.8}
$$

where formally $a(t, \widehat{x}, \widehat{\eta}) \sim \sum_{j=0}^{\infty} a_j(t, \widehat{x}, \widehat{\eta})$, $a_j(t, \widehat{x}, \widehat{\eta}) \in \mathscr{C}^\infty(\overline{\mathbb{R}}_+ \times T^*\widehat{D})$, $a_j(t, \widehat{x}, \widehat{\eta})$ is a quasi-homogeneous function of degree $-j$. The phase $\Psi(t, \widehat{x}, \widehat{\eta})$ should solve the eikonal equation

$$
\begin{gathered} \frac{\partial \Psi}{\partial t} - i\widehat{p}_0(\widehat{x}, \Psi'_{\widehat{x}}) = O(\left|\operatorname{Im} \Psi\right|^N), \forall N \geq 0, \\ \Psi|_{t=0} = \langle \widehat{x}, \widehat{\eta} \rangle \end{gathered} \tag{4.9}
$$

with $\operatorname{Im} \Psi \geq 0$, where $\widehat{p}_0$ denotes the principal symbol of $\Box_s^{(0)}$. From (4.6), we have

$$
\widehat{p}_0 = \sum_{j=1}^{n-1} \overline{\widehat{q}}_j \widehat{q}_j, \tag{4.10}
$$

where $\widehat{q}_j$ is the principal symbol of $Z_j - iZ_j(\phi)\frac{\partial}{\partial x_{2n}}$, $j = 1, \ldots, n-1$. The characteristic manifold $\widehat{\Sigma}$ of $\Box_s^{(0)}$ is given by

$$
\widehat{\Sigma} = \left\{ (\widehat{x}, \widehat{\xi}) \in T^*\widehat{D};\, \widehat{q}_1(\widehat{x}, \widehat{\xi}) = \cdots = \widehat{q}_{n-1}(\widehat{x}, \widehat{\xi}) = \overline{\widehat{q}}_1(\widehat{x}, \widehat{\xi}) = \cdots = \overline{\widehat{q}}_{n-1}(\widehat{x}, \widehat{\xi}) = 0 \right\}. \tag{4.11}
$$

From (4.11), we see that $\widehat{p}_0$ vanishes to second order at $\widehat{\Sigma}$. Let $\widehat{\sigma}$ denote the canonical two form on $T^*\widehat{D}$. As Proposition 3.2 and Theorem 3.3, we have

Theorem 4.3. *With the notations used above, we have*

$$
\widehat{\Sigma} = \left\{ (\widehat{x}, \widehat{\xi}) \in T^*\widehat{D};\, \widehat{\xi} = (\lambda\omega_0(x) - 2\mathrm{Im}\,\overline{\partial}_b\phi(x)\xi_{2n}, \xi_{2n}), \lambda \in \mathbb{R} \right\}. \tag{4.12}
$$

Put

$$
\begin{aligned} \widehat{\Sigma}_+ &= \left\{ (\widehat{x}, \widehat{\xi}) \in T^*\widehat{D};\, \widehat{\xi} = (\lambda\omega_0(x) - 2\mathrm{Im}\,\overline{\partial}_b\phi(x)\xi_{2n}, \xi_{2n}), \lambda \in \mathbb{R}, \xi_{2n} > 0 \right\}, \\ \widehat{\Sigma}_- &= \left\{ (\widehat{x}, \widehat{\xi}) \in T^*\widehat{D};\, \widehat{\xi} = (\lambda\omega_0(x) - 2\mathrm{Im}\,\overline{\partial}_b\phi(x)\xi_{2n}, \xi_{2n}), \lambda \in \mathbb{R}, \xi_{2n} < 0 \right\}. \end{aligned} \tag{4.13}
$$

Then, $\widehat{\sigma}$ is non-degenerate at every point of $\widehat{\Sigma}_+ \cup \widehat{\Sigma}_-$.

Put

$$
U = \left\{ (\widehat{x}, \widehat{\xi}) \in T^*\widehat{D};\, \widehat{\xi} = (\xi, \xi_{2n}), \xi_{2n} > 0 \right\}. \tag{4.14}
$$

Then U is a conic open set of $T^*\widehat{D}$. Until further notice, we work in U. Since $\widehat{\sigma}$ is non-degenerate at each point of $U\cap\widehat{\Sigma}=\widehat{\Sigma}_+$, (4.9) can be solved with $\operatorname{Im}\Psi\geq 0$ on U. More precisely, we have the following.

Theorem 4.4. *There exists $\Psi(t,\widehat{x},\widehat{\eta})\in\mathscr{C}^\infty(\overline{\mathbb{R}}_+\times U)$ such that $\Psi(t,\widehat{x},\widehat{\eta})$ is quasi-homogeneous of degree 1 and $\operatorname{Im}\Psi\geq 0$ and such that (4.9) holds where the error term is uniform on every set of the form $[0,T]\times K$ with $T>0$ and $K\subset U$ compact. Furthermore, Ψ is unique up to a term which is $O(\left|\operatorname{Im}\Psi\right|^N)$ locally uniformly for every N and*

$$\begin{gathered}\Psi(t,\widehat{x},\widehat{\eta})=\langle\widehat{x},\widehat{\eta}\rangle \text{ on } \widehat{\Sigma}_+,\\ d_{\widehat{x},\widehat{\eta}}(\Psi-\langle\widehat{x},\widehat{\eta}\rangle)=0 \text{ on } \widehat{\Sigma}_+.\end{gathered}\tag{4.15}$$

Moreover, we have

$$\operatorname{Im}\Psi(t,\widehat{x},\widehat{\eta})\asymp\Big(\left|\widehat{\eta}\right|\frac{t\left|\widehat{\eta}\right|}{1+t\left|\eta\right|}\Big)\Big(\operatorname{dist}\big((\widehat{x},\frac{\widehat{\eta}}{\left|\widehat{\eta}\right|}),\widehat{\Sigma}_+\big)\Big)^2,\quad t\geq 0,\quad(\widehat{x},\widehat{\eta})\in U.\tag{4.16}$$

Furthermore, we can take $\Psi(t,\widehat{x},\widehat{\eta})$ so that

$$\Psi(t,\widehat{x},\widehat{\eta})=\Psi(t,(x,0),\widehat{\eta})+x_{2n}\eta_{2n}.\tag{4.17}$$

Theorem 4.5. *There exists a function $\Psi(\infty,\widehat{x},\widehat{\eta})\in\mathscr{C}^\infty(U)$ with a uniquely determined Taylor expansion at each point of $\widehat{\Sigma}_+$ such that $\Psi(\infty,\widehat{x},\widehat{\eta})$ is positively homogeneous of degree 1 and for every compact set $K\subset U$ there is a $c_K>0$ such that $\operatorname{Im}\Psi(\infty,\widehat{x},\widehat{\eta})\geq c_K\left|\widehat{\eta}\right|\Big(\operatorname{dist}\big((\widehat{x},\frac{\widehat{\eta}}{\left|\widehat{\eta}\right|}),\widehat{\Sigma}_+\big)\Big)^2$, $d_{\widehat{x},\widehat{\eta}}(\Psi(\infty,\widehat{x},\widehat{\eta})-\langle\widehat{x},\widehat{\eta}\rangle)=0$ on $\widehat{\Sigma}_+$. If $\lambda\in C(U)$, $\lambda>0$ and $\lambda(\widehat{x},\widehat{\xi})<\min\lambda_j(\widehat{x},\widehat{\xi})$, for all $(\widehat{x},\widehat{\xi})=(\widehat{x},(\lambda\omega_0(x)-2\operatorname{Im}\overline{\partial}_b\phi(x)\xi_{2n},\xi_{2n}))\in\widehat{\Sigma}_+$, where $\lambda_j(\widehat{x},\widehat{\xi})$ are the eigenvalues of the Hermitian quadratic form $\xi_{2n}R^L_x$, then the solution $\Psi(t,\widehat{x},\widehat{\eta})$ of (4.9) can be chosen so that for every compact set $K\subset U$ and all indices α, β, γ, there is a constant $c_{\alpha,\beta,\gamma,K}>0$ such that*

$$\Big|\partial^\alpha_{\widehat{x}}\partial^\beta_{\widehat{\eta}}\partial^\gamma_t(\Psi(t,\widehat{x},\widehat{\eta})-\Psi(\infty,\widehat{x},\widehat{\eta}))\Big|\leq c_{\alpha,\beta,\gamma,K}e^{-\lambda(\widehat{x},\widehat{\eta})t} \text{ on } \overline{\mathbb{R}}_+\times K.\tag{4.18}$$

For the proofs of Theorem 4.4 and Theorem 4.5, we refer to Menikoff-Sjöstrand [23], [11] and [13, Section 4.1].

From now on, we assume that $\Psi(t,\widehat{x},\widehat{\eta})$ has the form (4.17) and hence

$$\Psi(\infty,\widehat{x},\widehat{\eta})=\Psi(\infty,(x,0),\widehat{\eta})+x_{2n}\eta_{2n}.\tag{4.19}$$

We let the full symbol of $\Box^{(0)}_s$ be:

$$\text{full symbol of } \Box^{(0)}_s=\sum_{j=0}^2\widehat{p}_j(\widehat{x},\widehat{\xi}),$$

where $\widehat{p}_j(\widehat{x},\widehat{\xi})$ is positively homogeneous of order $2-j$. We apply $\partial_t+\Box^{(0)}_s$ formally under the integral in (4.8) and then introduce the asymptotic expansion of $\Box^{(0)}_s(ae^{i\Psi})$. Setting $(\partial_t+\Box^{(0)}_s)(ae^{i\Psi})\sim 0$ and regrouping the terms according to the degree of quasi-homogeneity, we obtain for each N the transport equations

$$\begin{cases}T(t,\widehat{x},\widehat{\eta},\partial_t,\partial_{\widehat{x}})a_0=O(\left|\operatorname{Im}\Psi\right|^N),\\ T(t,\widehat{x},\widehat{\eta},\partial_t,\partial_{\widehat{x}})a_j+R_j(t,\widehat{x},\widehat{\eta},a_0,\ldots,a_{j-1})=O(\left|\operatorname{Im}\Psi\right|^N).\end{cases}\tag{4.20}$$

Here

$$T(t,\widehat{x},\widehat{\eta},\partial_t,\partial_{\widehat{x}})=\partial_t-i\sum_{j=1}^{2n}\frac{\partial\widehat{p}_0}{\partial\xi_j}(\widehat{x},\Psi'_{\widehat{x}})\frac{\partial}{\partial x_j}+q(t,\widehat{x},\widehat{\eta})\,,$$

where

$$q(t,\widehat{x},\widehat{\eta})=\widehat{p}_1(\widehat{x},\Psi'_{\widehat{x}})+\frac{1}{2i}\sum_{j,t=1}^{2n}\frac{\partial^2\widehat{p}_0(\widehat{x},\Psi'_{\widehat{x}})}{\partial\xi_j\partial\xi_t}\frac{\partial^2\Psi(t,\widehat{x},\widehat{\eta})}{\partial x_j\partial x_t}$$

and R_j is a linear differential operator acting on $a_0,a_1,\ldots,a_{j-1}$. We note that $q(t,\widehat{x},\widehat{\eta})\to q(\infty,\widehat{x},\widehat{\eta})$ as $t\to\infty$, exponentially fast in the sense of (4.18) and the same is true for the coefficients of R_j, for all j.

Following [13], we can solve the transport equations (4.20). To state the results precisely, we pause and introduce some symbol spaces.

Definition 4.6. Let $\mu\geq 0$ be a non-negative constant. We say that $a\in\widehat{S}^m_\mu(\overline{\mathbb{R}}_+\times U)$ if $a\in\mathscr{C}^\infty(\overline{\mathbb{R}}_+\times U)$ and for all indices $\alpha,\beta\in\mathbb{N}_0^{2n}$, $\gamma\in\mathbb{N}_0$, every compact set $K\Subset\widehat{D}$, there exists a constant $c>0$ such that

$$\left|\partial_t^\gamma\partial_{\widehat{x}}^\alpha\partial_{\widehat{\eta}}^\beta a(t,\widehat{x},\widehat{\eta})\right|\leq ce^{-t\mu|\eta_{2n}|}(1+|\eta|)^{m+\gamma-|\beta|},\ \widehat{x}\in K,(\widehat{x},\widehat{\eta})\in U.$$

Put

$$\widehat{S}^{-\infty}_\mu(\overline{\mathbb{R}}_+\times U):=\bigcap_{m\in\mathbb{R}}\widehat{S}^m_\mu(\overline{\mathbb{R}}_+\times U).$$

Let $a_j\in\widehat{S}^{m_j}_\mu(\overline{\mathbb{R}}_+\times U)$, $j\in\mathbb{N}_0$, with $m_j\to-\infty$, $j\to\infty$. Then there exists $a\in\widehat{S}^{m_0}_\mu(\overline{\mathbb{R}}_+\times U)$, unique modulo $\widehat{S}^{-\infty}_\mu(\overline{\mathbb{R}}_+\times U)$, such that $a-\sum\limits_{j=0}^{k-1}a_j\in\widehat{S}^{m_k}_\mu(\overline{\mathbb{R}}_+\times U)$ for $k=0,1,2,\ldots$. If a and a_j have the properties above, we write $a\sim\sum\limits_{j=0}^{\infty}a_j$ in $\widehat{S}^{m_0}_\mu(\overline{\mathbb{R}}_+\times U)$.

Following the proof of [13, Theorem 4.15] we get:

Theorem 4.7. *We can find solutions $a_j(t,\widehat{x},\widehat{\eta})\in\widehat{S}^{-j}_0(\overline{\mathbb{R}}_+\times U)$, $j=0,1,\ldots$ of the system* (4.20)*, where $a_j(t,\widehat{x},\widehat{\eta})$ is a quasi-homogeneous function of degree $-j$, for each j, with*

$$\begin{aligned}&a_0(0,\widehat{x},\widehat{\eta})=1 \text{ on } U,\\&a_j(t,\widehat{x},\widehat{\eta})=0 \text{ on } U,\ j=1,2,\ldots,\end{aligned}\tag{4.21}$$

$$\begin{aligned}&a_j(t,\widehat{x},\widehat{\eta})-a_j(\infty,\widehat{x},\widehat{\eta})\in\widehat{S}^{-j}_\mu(\overline{\mathbb{R}}_+\times U),\quad j=0,1,2,\ldots,\\&a_0(\infty,\widehat{x},\widehat{\eta})\neq 0,\ \ \forall(\widehat{x},\widehat{\eta})\in\widehat{\Sigma}_+,\end{aligned}\tag{4.22}$$

where $\mu>0$ is a constant and $a_j(\infty,\widehat{x},\widehat{\eta})\in\mathscr{C}^\infty(U)$, $j=0,1,\ldots$, $a_j(\infty,\widehat{x},\widehat{\eta})$ is a positively homogeneous function of degree $-j$, for each j.

Let $m\in\mathbb{R}$, $0\leq\rho,\delta\leq 1$. For a conic open subset Γ of $T^*\widehat{D}$, let $S^m_{\rho,\delta}(\Gamma)$ denote the Hörmander symbol space on Γ of order m type (ρ,δ) (see [8, Definition 1.1]) and let $S^m_{\rm cl}(\Gamma)$ denote the space of classical symbols on Γ of order m (see [8, p. 35]). Let $B\subset D$ be an open set. Let $L^m_{\frac{1}{2},\frac{1}{2}}(B)$ and $L^m_{\rm cl}(B)$ denote the space of pseudodifferential operators on B of order m type $(\frac{1}{2},\frac{1}{2})$ and the space of classical pseudodifferential operators on B of order m. The classical result of Calderon and Vaillancourt [10, Theorem 18.6.6] tells us that for any $A\in L^m_{\frac{1}{2},\frac{1}{2}}(B)$,

$$A:H^s_{\rm comp}(B)\to H^{s-m}_{\rm loc}(B)\text{ is continuous, for every }s\in\mathbb{R}.\tag{4.23}$$

We return to our situation. For $j \in \mathbb{N}_0$, let $a_j(t, \widehat{x}, \widehat{\eta}) \in \widehat{S}_0^{-j}(\overline{\mathbb{R}}_+ \times U)$ and $a_j(\infty, \widehat{x}, \widehat{\eta}) \in \mathscr{C}^\infty(U)$ be as in Theorem 4.7. Let

$$
\begin{aligned}
& a(\infty, \widehat{x}, \widehat{\eta}) \sim \sum_{j=0}^{\infty} a_j(\infty, \widehat{x}, \widehat{\eta}) \text{ in } S^0_{1,0}(U), \\
& a(t, \widehat{x}, \widehat{\eta}) \sim \sum_{j=0}^{\infty} a_j(t, \widehat{x}, \widehat{\eta}) \text{ in } \widehat{S}^0_0(\overline{\mathbb{R}}_+ \times U), \\
& a(t, \widehat{x}, \widehat{\eta}) - a(\infty, \widehat{x}, \widehat{\eta}) \in \widehat{S}^0_\mu(\overline{\mathbb{R}}_+ \times U), \quad \mu > 0.
\end{aligned}
\tag{4.24}
$$

Take $\alpha(\eta_{2n}) \in \mathscr{C}^\infty(\mathbb{R})$ with $\alpha(\eta_{2n}) = 1$ if $\eta_{2n} \leq \frac{1}{2}$, $\alpha(\eta_{2n}) = 0$ if $\eta_{2n} \geq 1$. Choose $\chi \in \mathscr{C}^\infty_0(\mathbb{R}^{2n})$ so that $\chi(\widehat{\eta}) = 1$ when $|\widehat{\eta}| < 1$ and $\chi(\widehat{\eta}) = 0$ when $|\widehat{\eta}| > 2$. For $\varepsilon > 0$, put

$$
\begin{aligned}
G_\varepsilon(\widehat{x}, \widehat{y}) = \frac{1}{(2\pi)^{2n}} \int \Big(\int_0^\infty & \big(e^{i(\Psi(t,\widehat{x},\widehat{\eta}) - \langle \widehat{y}, \widehat{\eta} \rangle)} a(t, \widehat{x}, \widehat{\eta}) \\
& - e^{i(\Psi(\infty,\widehat{x},\widehat{\eta}) - \langle \widehat{y}, \widehat{\eta} \rangle)} a(\infty, \widehat{x}, \widehat{\eta}) \big) (1 - \chi(\widehat{\eta})) \chi(\varepsilon \widehat{\eta}) (1 - \alpha(\eta_{2n})) dt \Big) d\widehat{\eta}.
\end{aligned}
$$

By Chapter 5 in part I of [11], we have for any $u \in \mathscr{C}^\infty_0(\widehat{D})$,

$$
\lim_{\varepsilon \to 0} \int G_\varepsilon(\widehat{x}, \widehat{y}) u(\widehat{y}) d\widehat{y} \in \mathscr{C}^\infty(\widehat{D}),
$$

and the operator

$$
G : \mathscr{C}^\infty_0(\widehat{D}) \to \mathscr{C}^\infty(\widehat{D}), \ \ u \mapsto \lim_{\varepsilon \to 0} \int G_\varepsilon(\widehat{x}, \widehat{y}) u(\widehat{y}) dy,
$$

is continuous, has a unique continuous extension: $G : \mathscr{E}'(\widehat{D}) \to \mathscr{D}'(\widehat{D})$ and $G \in L^{-1}_{\frac{1}{2},\frac{1}{2}}(\widehat{D})$ with symbol

$$
q(\widehat{x}, \widehat{\eta}) = \int_0^\infty \Big(e^{i(\Psi(t,\widehat{x},\widehat{\eta}) - \langle \widehat{x}, \widehat{\eta} \rangle)} a(t, \widehat{x}, \widehat{\eta}) - e^{i(\Psi(\infty,\widehat{x},\widehat{\eta}) - \langle \widehat{x}, \widehat{\eta} \rangle)} a(\infty, \widehat{x}, \widehat{\eta}) \Big) dt (1 - \alpha(\eta_{2n}))
$$

in $S^{-1}_{\frac{1}{2},\frac{1}{2}}(T^*\widehat{D})$. We denote

$$
\begin{aligned}
G(\widehat{x}, \widehat{y}) = \frac{1}{(2\pi)^{2n}} \int \Big(\int_0^\infty & \big(e^{i(\Psi(t,\widehat{x},\widehat{\eta}) - \langle \widehat{y}, \widehat{\eta} \rangle)} a(t, \widehat{x}, \widehat{\eta}) \\
& - e^{i(\Psi(\infty,\widehat{x},\widehat{\eta}) - \langle \widehat{y}, \widehat{\eta} \rangle)} a(\infty, \widehat{x}, \widehat{\eta}) \big) (1 - \chi(\widehat{\eta})) (1 - \alpha(\eta_{2n})) dt \Big) d\widehat{\eta}.
\end{aligned}
\tag{4.25}
$$

Similarly, for $\varepsilon > 0$, put

$$
S_\varepsilon(\widehat{x}, \widehat{y}) = \frac{1}{(2\pi)^{2n}} \int e^{i(\Psi(\infty,\widehat{x},\widehat{\eta}) - \langle \widehat{y}, \widehat{\eta} \rangle)} a(\infty, \widehat{x}, \widehat{\eta}) (1 - \chi(\widehat{\eta})) \chi(\varepsilon \widehat{\eta}) (1 - \alpha(\eta_{2n})) d\widehat{\eta}.
$$

By [11, Chapter 5, part I]) we have for $u \in \mathscr{C}^\infty_0(\widehat{D})$,

$$
\lim_{\varepsilon \to 0} \int S_\varepsilon(\widehat{x}, \widehat{y}) u(\widehat{y}) d\widehat{y} \in \mathscr{C}^\infty(\widehat{D}),
$$

the operator

$$
S : \mathscr{C}^\infty_0(\widehat{D}) \to \mathscr{C}^\infty(\widehat{D}), \ \ u \mapsto \lim_{\varepsilon \to 0} \int S_\varepsilon(\widehat{x}, \widehat{y}) u(\widehat{y}) dy,
$$

is continuous, has a unique continuous extension: $S : \mathscr{E}'(\widehat{D}) \to \mathscr{D}'(\widehat{D})$ and $S \in L^0_{\frac{1}{2},\frac{1}{2}}(\widehat{D})$ with symbol $s(\widehat{x},\widehat{\eta}) = e^{i(\Psi(\infty,\widehat{x},\widehat{\eta})-\langle\widehat{x},\widehat{\eta}\rangle)}a(\infty,\widehat{x},\widehat{\eta})(1-\alpha(\eta_{2n})) \in S^0_{\frac{1}{2},\frac{1}{2}}(T^*\widehat{D})$. We denote

$$S(\widehat{x},\widehat{y}) = \frac{1}{(2\pi)^{2n}}\int e^{i(\Psi(\infty,\widehat{x},\widehat{\eta})-\langle\widehat{y},\widehat{\eta}\rangle)}a(\infty,\widehat{x},\widehat{\eta})\big(1-\chi(\widehat{\eta})\big)\big(1-\alpha(\eta_{2n})\big)d\widehat{\eta}. \tag{4.26}$$

Put

$$\widetilde{I} = (2\pi)^{-2n}\int e^{i\langle\widehat{x}-\widehat{y},\widehat{\eta}\rangle}\big(1-\alpha(\eta_{2n})\big)d\widehat{\eta}. \tag{4.27}$$

We can repeat the proof of [11, Proposition 6.5] with minor changes and obtain:

Theorem 4.8. *With the notations used above, we have*

$$\begin{aligned}
&S + \Box^{(0)}_s \circ G \equiv \widetilde{I} \ \text{ on } \widehat{D},\\
&\overline{\partial}_s \circ S \equiv 0 \ \text{ on } \widehat{D},\ \ \Box^{(0)}_s \circ S \equiv 0 \ \text{ on } \widehat{D}.
\end{aligned}$$

The next result follows from complex stationary phase formula of Melin and Sjöstrand [22] with essentially the same proof as of [13, Theorem 4.29].

Theorem 4.9. *With the notations and assumptions above, let $S = S(\widehat{x},\widehat{y}) \in L^0_{\frac{1}{2},\frac{1}{2}}(\widehat{D})$ be as in Theorem 4.8. Then, on $\widehat{D}$, we have*

$$S(\widehat{x},\widehat{y}) \equiv \int_{u\in\mathbb{R},t\in\mathbb{R}_+} e^{i\Phi(\widehat{x},\widehat{y},u,t)}b(\widehat{x},\widehat{y},u,t)(1-\alpha(t))dudt \tag{4.28}$$

with

$$\begin{aligned}
&b(\widehat{x},\widehat{y},u,t) \sim \sum_{j=0}^{\infty} b_j(\widehat{x},\widehat{y},u,t) \text{ in } S^{n-1}_{1,0}(\widehat{D}\times\widehat{D}\times\mathbb{R}\times\mathbb{R}_+),\\
&b_j(\widehat{x},\widehat{y},u,t) \in \mathscr{C}^\infty(\widehat{D}\times\widehat{D}\times\mathbb{R}\times\mathbb{R}_+),\ \ j=0,1,2,\ldots,\\
&b_j(\widehat{x},\widehat{y},\lambda u,\lambda t) = \lambda^{n-1-j}b_j(\widehat{x},\widehat{y},u,t),\ \ \forall(\widehat{x},\widehat{y},u,t)\in\widehat{D}\times\widehat{D}\times\mathbb{R}\times\mathbb{R}_+,\ \ \lambda\geq 1,\ \ \forall j,\\
&b_0(\widehat{x},\widehat{x},u,t) \neq 0,\ \ \forall(\widehat{x},\widehat{y},u,t)\in\widehat{D}\times\widehat{D}\times\mathbb{R}\times\mathbb{R}_+,\ \ \lambda\geq 1,
\end{aligned} \tag{4.29}$$

$$\begin{aligned}
&\Phi(\widehat{x},\widehat{y},u,t) = (x_{2n}-y_{2n})t + \varphi(x,y,u,t),\\
&\varphi(x,y,u,t) \in \mathscr{C}^\infty(D\times D\times\mathbb{R}\times\mathbb{R}_+),\\
&\varphi(x,y,\lambda u,\lambda t) = \lambda\varphi(x,y,u,t),\ \ \forall(x,y,u,t)\in D\times D\times\mathbb{R}\times\mathbb{R}_+,\ \ \lambda\geq 1,\\
&\operatorname{Im}\varphi(x,y,u,t) \geq 0,\\
&\varphi(x,x,u,t) = 0,\ \ \forall x\in D,\ \ u\in\mathbb{R},\ \ t\in\mathbb{R}_+,\\
&d_x\varphi|_{(x,x,u,t)} = -2t\operatorname{Im}\overline{\partial}_b\phi(x) + u\omega_0(x),\ \ \forall x\in D,\ \ u\in\mathbb{R},\ \ t\in\mathbb{R}_+,\\
&d_y\varphi|_{(x,x,u,t)} = 2t\operatorname{Im}\overline{\partial}_b\phi(x) - u\omega_0(x),\ \ \forall x\in D,\ \ u\in\mathbb{R},\ \ t\in\mathbb{R}_+,\\
&\tfrac{\partial\varphi}{\partial u}(x,y,u,t) = 0 \text{ and } \tfrac{\partial\varphi}{\partial t}(x,y,u,t) = 0 \text{ if and only if } x = y.
\end{aligned} \tag{4.30}$$

We can repeat the method in [13, Section 4.4] with minor changes to compute the tangential Hessian of the phase function $\varphi(x,y,u,t)$. Since the computation is simpler therefore we omit the details. We state the result.

Theorem 4.10. *With the notations above, put $\psi(x,y,u) := \varphi(x,y,u,1)$. Fix $p \in D$ and let $\overline{Z}_1,\ldots,\overline{Z}_{n-1}$ be an orthonormal frame of $T^{1,0}_xX$ varying smoothly with x in a neighbourhood of p, for which the Hermitian quadratic form R^L_x is diagonalized at p. Let $x=(x_1,\ldots,x_{2n-1})$, $z_j = x_{2j-1}+ix_{2j}$, $j=1,\ldots,n-1$, be local coordinates of X defined in some small neighbourhood of p such that*

$$
\begin{aligned}
&x(p)=0,\ \ \omega_0(p)=dx_{2n-1},\ \ T(p)=-\frac{\partial}{\partial x_{2n-1}}(p),\\
&\Big\langle \frac{\partial}{\partial x_j}(p)\,|\,\frac{\partial}{\partial x_t}(p)\Big\rangle = 2\delta_{j,t},\ \ j,t=1,\ldots,2n-2,\\
&\overline{Z}_j(p)=\frac{\partial}{\partial z_j}+i\sum_{t=1}^{n-1}\tau_{j,t}\overline{z}_t\frac{\partial}{\partial x_{2n-1}}+c_jx_{2n-1}\frac{\partial}{\partial x_{2n-1}}+O(|x|^2),\ \ j=1,\ldots,n-1,\\
&\phi(x)=\beta x_{2n-1}+\sum_{j=1}^{n-1}\big(\alpha_jz_j+\overline{\alpha}_j\overline{z}_j\big)+\frac{1}{2}\sum_{l,t=1}^{n-1}\mu_{t,l}z_t\overline{z}_l+\sum_{l,t=1}^{n-1}\big(a_{l,t}z_lz_t+\overline{a}_{l,t}\overline{z}_l\overline{z}_t\big)\\
&\quad+\sum_{j=1}^{n-1}\big(d_jz_jx_{2n-1}+\overline{d}_j\overline{z}_jx_{2n-1}\big)+O(|x_{2n-1}|^2)+O(|x|^3),
\end{aligned}
\tag{4.31}
$$

where $\beta\in\mathbb{R}$, $\tau_{j,t}, c_j, \alpha_j, \mu_{j,t}, a_{j,t}, d_j \in \mathbb{C}$, $\mu_{j,t}=\overline{\mu}_{t,j}$, $\tau_{j,t}+\overline{\tau}_{t,j}=0$, $j,t=1,\ldots,n-1$. We also write $y=(y_1,\ldots,y_{2n-1})$, $w_j=y_{2j-1}+iy_{2j}$, $j=1,\ldots,n-1$. Then, in some small neighbourhood D_0 of p we have for all $(x,y,u)\in D_0\times D_0\times\mathbb{R}$,

$$
\begin{aligned}
&\operatorname{Im}\psi(x,y,u)\geq c\,|x'-y'|^2\,,\\
&\operatorname{Im}\psi(x,y,u)+\left|\frac{\partial\psi}{\partial u}(x,y,u)\right|\geq c(|x_{2n-1}-y_{2n-1}|+|x'-y'|^2),
\end{aligned}
\tag{4.32}
$$

where $c>0$ is a constant, $x'=(x_1,\ldots,x_{2n-2})$, $y'=(y_2,\ldots,y_{2n-2})$, $|x'-y'|^2=\sum\limits_{j=1}^{2n-2}|x_j-y_j|^2$ and

$$
\begin{aligned}
&\psi(x,y,u)\\
&=-i\sum_{j=1}^{n-1}\alpha_j(z_j-w_j)+i\sum_{j=1}^{n-1}\overline{\alpha}_j(\overline{z}_j-\overline{w}_j)+u(x_{2n-1}-y_{2n-1})-\frac{i}{2}\sum_{j,l=1}^{n-1}(a_{l,j}+a_{j,l})(z_jz_l-w_jw_l)\\
&\quad+\frac{i}{2}\sum_{j,l=1}^{n-1}(\overline{a}_{l,j}+\overline{a}_{j,l})(\overline{z}_j\overline{z}_l-\overline{w}_j\overline{w}_l)+\frac{1}{2}\sum_{j,l=1}^{n-1}iu(\overline{\tau}_{l,j}-\tau_{j,l})(z_j\overline{z}_l-w_j\overline{w}_l)\\
&\quad+\sum_{j=1}^{n-1}(-ic_j\beta-uc_j-id_j)(z_jx_{2n-1}-w_jy_{2n-1})+\sum_{j=1}^{n-1}(i\overline{c}_j\beta-u\overline{c}_j+i\overline{d}_j)(\overline{z}_jx_{2n-1}-\overline{w}_jy_{2n-1})\\
&\quad-\frac{i}{2}\sum_{j=1}^{n-1}\lambda_j(z_j\overline{w}_j-\overline{z}_jw_j)+\frac{i}{2}\sum_{j=1}^{n-1}\lambda_j\,|z_j-w_j|^2+(x_{2n-1}-y_{2n-1})f(x,y,u)+O(|(x,y)|^3),\\
&\qquad f\in\mathscr{C}^\infty,\ \ f(0,0,u)=0,\ \ \forall u\in\mathbb{R},
\end{aligned}
\tag{4.33}
$$

where $\lambda_1>0,\ldots,\lambda_{n-1}>0$ are the eigenvalues of R^L_p with respect to $\langle\cdot|\cdot\rangle$.

4.2. **Semi-classical Hodge decomposition for $\Box^{(0)}_{s,k}$.** In this section we apply Theorem 4.8 and Theorem 4.9 to describe the semi-classical behaviour of $\Box^{(0)}_{s,k}$.

Let s be a local trivializing section of L on an open subset $D \subset X$ and $|s|^2_h = e^{-2\phi}$. Let $\chi(x_{2n}), \chi_1(x_{2n}) \in \mathscr{C}^\infty_0(\mathbb{R})$, $\chi, \chi_1 \geq 0$. We assume that $\chi_1 = 1$ on $\operatorname{Supp}\chi$. We take χ so that $\int \chi(x_{2n})dx_{2n} = 1$. Put

$$\chi_k(x_{2n}) = e^{ikx_{2n}}\chi(x_{2n}). \tag{4.34}$$

We say that a sequence (g_k) in $\mathbb{C}$ is rapidly decreasing and write $g_k = O(k^{-\infty})$ if for every $N > 0$, there exists $C_N > 0$ independent of k such that for all k we have $|g_k| \leq C_N k^{-N}$.

Proposition 4.11. *With the notations before, let $\widetilde{I} = (2\pi)^{-2n}\int e^{i\langle\widehat{x}-\widehat{y},\widehat{\eta}\rangle}(1-\alpha(\eta_{2n}))d\widehat{\eta}$ be as in (4.27). Let $\widetilde{I}_k$ be the continuous operator $\mathscr{C}^\infty_0(D) \to \mathscr{C}^\infty(D)$ given by*

$$\widetilde{I}_k : \mathscr{C}^\infty_0(D) \to \mathscr{C}^\infty(D),\ \ f \longmapsto \int e^{-ikx_{2n}}\chi_1(x_{2n})\widetilde{I}(\chi_k f)(\widehat{x})dx_{2n}.$$

Then, $\widetilde{I}_k = (1+g_k)I$ on $\mathscr{C}^\infty_0(D)$, where I is the identity map on $\mathscr{C}^\infty_0(D)$ and (g_k) is a rapidly decreasing sequence.

Proof. It is easy to see that

$$I = (2\pi)^{-2n}\int e^{i\langle\widehat{x}-\widehat{y},\widehat{\eta}\rangle - ik(x_{2n}-y_{2n})}\chi_1(x_{2n})\chi(y_{2n})d\widehat{\eta}dy_{2n}dx_{2n} \text{ on } \mathscr{C}^\infty_0(D).$$

From this observation, we can check that $\widetilde{I}_k = (1+g_k)I$ where

$$g_k = -(2\pi)^{-2n}\int e^{i\langle x_{2n}-y_{2n},\eta_{2n}-k\rangle}\alpha(\eta_{2n})\chi_1(x_{2n})\chi(y_{2n})d\eta_{2n}dy_{2n}dx_{2n}. \tag{4.35}$$

Since $\alpha(\eta_{2n}) = 0$ if $\eta \geq 1$, we can integrate by parts in (4.35) with respect to y_{2n} several times and conclude that $g_k = O(k^{-\infty})$. ☐

Let $S \in L^0_{\frac{1}{2},\frac{1}{2}}(\widehat{D})$ and $G \in L^{-1}_{\frac{1}{2},\frac{1}{2}}(\widehat{D})$ be as in Theorem 4.8. For $s \in \mathbb{N}_0$ define

$$\mathcal{S}_k : H^s_{\mathrm{comp}}(D) \to H^s_{\mathrm{loc}}(D),\ \ f \longmapsto \frac{1}{1+g_k}\int e^{-ikx_{2n}}\chi_1(x_{2n})S(\chi_k f)(\widehat{x})dx_{2n}\,, \tag{4.36}$$

$$\mathcal{G}_k : H^s_{\mathrm{loc}}(D) \to H^{s+1}_{\mathrm{loc}}(D),\ \ f \longmapsto \frac{1}{1+g_k}\int e^{-ikx_{2n}}\chi_1(x_{2n})G(\chi_k f)(\widehat{x})dx_{2n}\,. \tag{4.37}$$

From (4.36), (4.37) and the fact that $S : H^s_{\mathrm{comp}}(\widehat{D}) \to H^s_{\mathrm{loc}}(\widehat{D})$ is continuous for every $s \in \mathbb{R}$, $G : H^s_{\mathrm{comp}}(\widehat{D}) \to H^{s+1}_{\mathrm{loc}}(\widehat{D})$ is continuous for every $s \in \mathbb{R}$, it is straightforward to check that

$$\begin{aligned}&\mathcal{S}_k = O(k^s) : H^s_{\mathrm{comp}}(D) \to H^s_{\mathrm{loc}}(D),\ \ \forall s \in \mathbb{N}_0,\\ &\mathcal{G}_k = O(k^s) : H^s_{\mathrm{comp}}(D) \to H^{s+1}_{\mathrm{loc}}(D),\ \ \forall s \in \mathbb{N}_0.\end{aligned} \tag{4.38}$$

Repeating the proof of [13, Theorem 5.4] by making use of Proposition 4.11 we get:

Theorem 4.12. *Let s be a local trivializing section of L on an open subset $D \subset X$ and $|s|^2_h = e^{-2\phi}$. Let $\mathcal{S}_k$ and $\mathcal{G}_k$ be as in (4.36), (4.37) respectively. Then,*

$$\begin{aligned}&\mathcal{S}^*_k, \mathcal{S}_k = O(k^s) : H^s_{\mathrm{comp}}(D) \to H^s_{\mathrm{loc}}(D),\ \ \forall s \in \mathbb{Z},\\ &\mathcal{G}^*_k, \mathcal{G}_k = O(k^s) : H^s_{\mathrm{comp}}(D) \to H^{s+1}_{\mathrm{loc}}(D),\ \ \forall s \in \mathbb{Z},\end{aligned} \tag{4.39}$$

and we have on D,

$$\overline{\partial}_{s,k}\mathcal{S}_k \equiv 0 \mod O(k^{-\infty})\,, \tag{4.40}$$

$$\Box^{(0)}_{s,k}\mathcal{S}_k \equiv 0,\ \ \mathcal{S}^*_k\Box^{(0)}_{s,k} \equiv 0 \mod O(k^{-\infty})\,, \tag{4.41}$$

$$\mathcal{S}_k + \Box^{(0)}_{s,k}\mathcal{G}_k = I\,, \tag{4.42}$$

$$\mathcal{G}^*_k\Box^{(0)}_{s,k} + \mathcal{S}^*_k = I\,, \tag{4.43}$$

where $\mathcal{S}^*_k$, $\mathcal{G}^*_k$ are the formal adjoints of $\mathcal{S}_k$, $\mathcal{G}_k$ with respect to $(\,\cdot\,|\,\cdot\,)$ respectively and $\Box^{(0)}_{s,k}$ is given by (3.3).

Theorem 4.13. *We have*

$$\mathcal{S}_k(x,y) - \int e^{ik\psi(x,y,u)}s(x,y,u,k)du = O(k^{-\infty}) : H^s_{\rm comp}(D) \to H^s_{\rm loc}(D),\ \forall s\in\mathbb{Z}, \tag{4.44}$$

where

$$\begin{aligned}&s(x,y,u,k)\in S^n_{\rm loc\,,cl}(1;D\times D\times\mathbb{R}),\\ &s(x,y,u,k)\sim\sum_{j=0}^{\infty}s_j(x,y,u)k^{n-j} \text{ in } S^n_{\rm loc}(1;D\times D\times\mathbb{R}),\\ &s_j(x,y,u)\in\mathscr{C}^\infty(D\times D\times\mathbb{R}),\ \ j=0,1,2,\ldots,\end{aligned} \tag{4.45}$$

and $\psi(x,y,u)=\varphi(x,y,u,1)$, $\varphi(x,y,u,t)$ is as in Theorem 4.9.

Proof. From the definition (4.36) of $\mathcal{S}_k$ and Theorem 4.9, we see that the distribution kernel of $\mathcal{S}_k$ is given by

$$\begin{aligned}&\mathcal{S}_k(x,y)\\ &\equiv\int_{t\geq0}e^{i\Phi(\widehat{x},\widehat{y},u,t)-ikx_{2n}+iky_{2n}}b(\widehat{x},\widehat{y},u,t)\chi_1(x_{2n})\chi(y_{2n})(1-\alpha(t))dx_{2n}dtdy_{2n}du \mod O(k^{-\infty})\\ &\equiv\int_{u\in\mathbb{R},\sigma\in\mathbb{R}_+}e^{ik\sigma\psi(x,y,u)+ik(x_{2n}-y_{2n})(\sigma-1)}k^2\sigma b(\widehat{x},\widehat{y},k\sigma u,k\sigma)\\ &\qquad\times\chi_1(x_{2n})\chi(y_{2n})(1-\alpha(k\sigma))dx_{2n}d\sigma dy_{2n}du \mod O(k^{-\infty}),\end{aligned} \tag{4.46}$$

where the integrals above are defined as oscillatory integrals. Let $\gamma(\sigma)\in\mathscr{C}^\infty_0(\mathbb{R}_+)$ with $\gamma(\sigma)=1$ in some small neighbourhood of 1. Put

$$\begin{aligned}&I_0(x,y)\\ &:=\int_{\sigma\geq0}e^{ik\sigma\psi(x,y,u)+ik(x_{2n}-y_{2n})(\sigma-1)}\gamma(\sigma)k^2\sigma b(\widehat{x},\widehat{y},k\sigma u,k\sigma)(1-\alpha(k\sigma))\\ &\qquad\times\chi_1(x_{2n})\chi(y_{2n})dx_{2n}d\sigma dy_{2n}du,\end{aligned} \tag{4.47}$$

$$\begin{aligned}&I_1(x,y)\\ &:=\int_{\sigma\geq0}e^{ik\sigma\psi(x,y,u)+ik(x_{2n}-y_{2n})(\sigma-1)}(1-\gamma(\sigma))k^2\sigma b(\widehat{x},\widehat{y},k\sigma u,k\sigma)(1-\alpha(k\sigma))\\ &\qquad\times\chi_1(x_{2n})\chi(y_{2n})dx_{2n}d\sigma dy_{2n}du.\end{aligned} \tag{4.48}$$

Then,

$$\mathcal{S}_k(x,y)\equiv I_0(x,y)+I_1(x,y) \mod O(k^{-\infty}). \tag{4.49}$$

First, we study $I_1(x,y)$. Note that when $\sigma\neq 1$, $d_{y_{2n}}\big(\sigma\psi(x,y,u)+(x_{2n}-y_{2n})(\sigma-1)\big)=1-\sigma\neq 0$. Thus, we can integrate by parts in y_{2n} several times and get that

$$I_1=O(k^{-\infty}):H^s_{\rm comp}(D)\to H^s_{\rm loc}(D),\ \ \forall s\in\mathbb{Z}. \tag{4.50}$$

Next, we study the kernel $I_0(x,y)$. We may assume that $b(\widehat{x},\widehat{y},k\sigma u,k\sigma)$ is supported in some small neighbourhood of $\widehat{x}=\widehat{y}$. We want to apply the stationary phase method of Melin and Sjöstrand (see page 148 of[22]) to carry out the $dx_{2n}d\sigma$ integration in (4.47). Put

$$\Theta(\widehat{x},\widehat{y},\sigma):=\sigma\psi(x,y,u)+(x_{2n}-y_{2n})(\sigma-1).$$

We first notice that $d_\sigma\Theta(\widehat{x},\widehat{y},\sigma)|_{\widehat{x}=\widehat{y}}=0$ and $d_{x_{2n}}\Theta(\widehat{x},\widehat{y},\sigma)|_{\sigma=1}=0$. Thus, $x=y$ and $\sigma=1$ are real critical points. Moreover, we can check that the Hessian of $\Theta(\widehat{x},\widehat{y},\sigma)$ at $\widehat{x}=\widehat{y}$, $\sigma=1$, is given by

$$\left(\begin{array}{cc}\Theta''_{\sigma\sigma}(\widehat{x},\widehat{x},1) & \Theta''_{x_{2n}\sigma}(\widehat{x},\widehat{x},1)\\ \Theta''_{\sigma x_{2n}}(\widehat{x},\widehat{x},1) & \Theta''_{x_{2n}x_{2n}}(\widehat{x},\widehat{x},1)\end{array}\right)=\left(\begin{array}{cc}0&1\\1&0\end{array}\right).$$

Thus, $\Theta(\widehat{x},\widehat{y},\sigma)$ is a non-degenerate complex valued phase function in the sense of Melin-Sjöstrand [22]. Let

$$\widetilde{\Theta}(\widetilde{\widehat{x}},\widetilde{\widehat{y}},\widetilde{\sigma}):=\widetilde{\psi}(\widetilde{x},\widetilde{y},u)\widetilde{\sigma}+(\widetilde{x}_{2n}-\widetilde{y}_{2n})(\widetilde{\sigma}-1)$$

be an almost analytic extension of $\Theta(\widehat{x},\widehat{y},\widehat{\sigma})$, where $\widetilde{\psi}(\widetilde{x},\widetilde{y},u)$ is an almost analytic extension of $\psi(x,y,u)$. Here we fix u. We can check that given y_{2n} and (x,y), $\widetilde{x}_{2n}=y_{2n}-\psi(x,y,u)$, $\widetilde{\sigma}=1$ are the solutions of

$$\frac{\partial\widetilde{\Theta}}{\partial\widetilde{\sigma}}=0\,,\ \frac{\partial\widetilde{\Theta}}{\partial\widetilde{x}_{2n}}=0.$$

From this and by the stationary phase formula of Melin-Sjöstrand [22], we get

$$I_0(x,y)-\int e^{ik\psi(x,y,u)}s(x,y,u,k)du=O(k^{-\infty}):H^s_{\rm comp}(D)\to H^s_{\rm loc}(D),\ \ \forall s\in\mathbb{Z}, \tag{4.51}$$

where $s(x,y,u,k)\in S^n_{{\rm loc},{\rm cl}}(1,D\times D\times\mathbb{R})$,

$$s(x,y,u,k)\sim\sum_{j=0}^\infty s_j(x,y,u)k^{n-j}\ \text{ in } S^n_{\rm loc}(1,D\times D\times\mathbb{R}),$$

$s_j(x,y,u)\in\mathscr{C}^\infty(D\times D\times\mathbb{R})$, $j=0,1,2,\ldots$. From (4.50), (4.51) and (4.49), the theorem follows. □

From Theorem 4.13 and the stationary phase method of Melin and Sjöstrand, we deduce:

Theorem 4.14. *Let $\mathcal{A}_k$ be a properly supported classical semi-classical pseudodifferential operator on D of order 0 as in* (2.8) *and* (2.9) *with symbol $\beta\in S^0_{{\rm loc},{\rm cl}}(1;T^*D)$ such that $\beta(x,\eta,k)=0$ if $|\eta|\geq\frac{1}{2}M$, for some large $M>0$. We have*

$$(\mathcal{S}_k\circ\mathcal{A}_k)(x,y)\equiv\int e^{ik\psi(x,y,u)}a(x,y,u,k)du\mod O(k^{-\infty}), \tag{4.52}$$

where

$$a(x,y,u,k)\in\mathscr{C}^\infty_0(D\times D\times(-M,M))\cap S^n_{{\rm loc},{\rm cl}}(1;D\times D\times(-M,M)),$$

$$a(x,y,u,k)\sim\sum_{j=0}^\infty a_j(x,y,u)k^{n-j}\ \textit{in}\ S^n_{\rm loc}(1;D\times D\times-]M,M[), \tag{4.53}$$

$$a_j(x,y,u)\in\mathscr{C}^\infty_0(D\times D\times(-M,M)),\ \ j=0,1,2,\ldots,$$

and $\psi(x,y,u)=\varphi(x,y,u,1)$*,* $\varphi(x,y,u,t)$ *is as in Theorem 4.9.*

Let $\mathcal{A}_k\equiv\frac{k^{2n-1}}{(2\pi)^{2n-1}}\int e^{ik\langle x-y,\eta\rangle}\beta(x,\eta,k)d\eta \mod O(k^{-\infty})$ be as in Theorem 4.14. Put

$$\beta(x,\eta,k)\sim\sum_{j=0}^{\infty}\beta_j(x,\eta)k^{-j},\ \ \beta_j(x,\eta)\in\mathscr{C}^\infty(T^*D),\ \ j=0,1,2,\ldots. \tag{4.54}$$

From the last formula of (4.29), it is straightforward to see that

$$a_0(x,x,u)\neq 0\ \text{ if } \beta_0(x,u\omega_0(x)-2\mathrm{Im}\,\overline{\partial}_b\phi(x))\neq 0, \tag{4.55}$$

where $a_0(x,y,u)$ is as in (4.53). In the rest of this section, we will calculate $a_0(x,x,u)$.

Fix $D_0\Subset D$ and let $\chi,\widehat{\chi}\in\mathscr{C}^\infty_0(D,[0,1])$, $\chi=\widehat{\chi}=1$ on D_0 and $\chi=1$ on some neighbourhood of $\mathrm{Supp}\,\widehat{\chi}$.

Lemma 4.15. *With the notations above, we have*

$$(\widehat{\chi}\mathcal{A}^*_k\mathcal{S}^*_k\chi)(\chi\mathcal{S}_k\mathcal{A}_k\widehat{\chi})\equiv\widehat{\chi}\mathcal{A}^*_k\mathcal{S}_k\mathcal{A}_k\widehat{\chi}\mod O(k^{-\infty}), \tag{4.56}$$

where $\mathcal{A}^*_k$ *is the formal adjoint of* $\mathcal{A}_k$*.*

Proof. From (4.43), we have

$$\widehat{\chi}\mathcal{A}^*_k\mathcal{G}^*_k\Box^{(0)}_{s,k}\chi+\widehat{\chi}\mathcal{A}^*_k\mathcal{S}^*_k\chi=\widehat{\chi}\mathcal{A}^*_k\chi. \tag{4.57}$$

From (4.57), we have

$$\widehat{\chi}\mathcal{A}^*_k\mathcal{G}^*_k\Box^{(0)}_{s,k}\chi^2\mathcal{S}_k\mathcal{A}_k\widehat{\chi}+\widehat{\chi}\mathcal{A}^*_k\mathcal{S}^*_k\chi^2\mathcal{S}_k\mathcal{A}_k\widehat{\chi}=\widehat{\chi}\mathcal{A}^*_k\chi^2\mathcal{S}_k\mathcal{A}_k\widehat{\chi}. \tag{4.58}$$

From (4.52), it is not difficult to check that $\mathcal{S}_k\mathcal{A}_k$ is k-negligible away the diagonal. From this observation, (4.39) and (4.41), we conclude that

$$\widehat{\chi}\mathcal{A}^*_k\mathcal{G}^*_k\Box^{(0)}_{s,k}\chi^2\mathcal{S}_k\mathcal{A}_k\widehat{\chi}\equiv 0\mod O(k^{-\infty}). \tag{4.59}$$

From (4.59) and (4.58), we get

$$\widehat{\chi}\mathcal{A}^*_k\mathcal{S}^*_k\chi^2\mathcal{S}_k\mathcal{A}_k\widehat{\chi}\equiv\widehat{\chi}\mathcal{A}^*_k\chi^2\mathcal{S}_k\mathcal{A}_k\widehat{\chi}\mod O(k^{-\infty}). \tag{4.60}$$

Again, since $\mathcal{S}_k\mathcal{A}_k$ is k-negligible away the diagonal, we deduce that

$$\widehat{\chi}\mathcal{A}^*_k\chi^2\mathcal{S}_k\mathcal{A}_k\widehat{\chi}\equiv\widehat{\chi}\mathcal{A}^*_k\mathcal{S}_k\mathcal{A}_k\widehat{\chi}\mod O(k^{-\infty}). \tag{4.61}$$

From (4.60) and (4.61), we get (4.56). □

From (4.56), (4.52) and the complex stationary phase formula of Melin-Sjöstrand [22], we deduce that

$$\big((\widehat{\chi}\mathcal{A}^*_k\mathcal{S}^*_k\chi)(\chi\mathcal{S}_k\mathcal{A}_k\widehat{\chi})\big)(x,y)\equiv(\widehat{\chi}\mathcal{A}^*_k\mathcal{S}_k\mathcal{A}_k\widehat{\chi})(x,y)\equiv\int e^{ik\psi(x,y,u)}g(x,y,u,k)du\mod O(k^{-\infty}), \tag{4.62}$$

where

$$\begin{aligned}&g(x,y,u,k)\in\mathscr{C}^\infty_0(D\times D\times(-M,M))\cap S^n_{\mathrm{loc,cl}}(1;D\times D\times(-M,M)),\\&g(x,y,u,k)\sim\sum_{j=0}^{\infty}g_j(x,y,u)k^{n-j}\text{ in } S^n_{\mathrm{loc}}(1;D\times D\times\mathbb{R}),\\&g_j(x,y,u)\in\mathscr{C}^\infty_0(D\times D\times(-M,M)),\ \ j=0,1,2,\ldots,\end{aligned} \tag{4.63}$$

and

$$g_0(x,x,u)=a_0(x,x,u)\overline{\beta_0}(x,u\omega_0(x)-2\mathrm{Im}\,\overline{\partial}_b\phi(x)),\ \ \forall(x,x,u)\in D_0\times D_0\times(-M,M). \tag{4.64}$$

On the other hand, we can repeat the procedure of Section 5 in [13] (see the discussion after Theorem 5.6 in [13]) and deduce that

$$\big((\widehat{\chi}\mathcal{A}_k^*\mathcal{S}_k^*\chi)(\chi\mathcal{S}_k\mathcal{A}_k\widehat{\chi})\big)(x,y)\equiv\int e^{ik\psi_1(x,y,u)}h(x,y,u,k)du \mod O(k^{-\infty}) \tag{4.65}$$

with

$$\begin{aligned}&h(x,y,u,k)\in S^n_{\mathrm{loc\,,cl}}(1,D\times D\times(-M,M))\cap\mathscr{C}_0^\infty(D\times D\times(-M,M)),\\&h(x,y,u,k)\sim\sum_{j=0}^\infty h_j(x,y,u)k^{n-j}\text{ in }S^n_{\mathrm{loc}}(1,D\times D\times(-M,M)),\\&h_j(x,y,u)\in\mathscr{C}_0^\infty(D\times D\times(-M,M)),\ \ j=0,1,2,\ldots,\end{aligned}\tag{4.66}$$

$$\begin{aligned}&h_0(x,x,u)=2\pi^n\left|\det R_x^L\right|^{-1}|a_0(x,x,u)|^2,\ \ \forall(x,x,u)\in D_0\times D_0\times(-M,M),\\&g_0(x,x,u)=h_0(x,x,u),\ \ \forall(x,x,u)\in D\times D\times(-M,M),\end{aligned}\tag{4.67}$$

and for all $(x,x,u)\in D\times D\times(-M,M)$, we have

$$\begin{aligned}&\psi_1(x,x,u)=0,\ \ d_x\psi_1(x,x,u)=d_x\psi(x,x,u),\ \ d_y\psi_1(x,x,u)=d_y\psi(x,x,u),\\&\operatorname{Im}\psi_1(x,y,u)\geq0,\ \ \forall(x,y,u)\in D\times D\times(-M,M).\end{aligned}\tag{4.68}$$

From (4.67) and (4.64), we get for all $(x,x,u)\in D_0\times D_0\times(-M,M)$,

$$a_0(x,x,u)\overline{\beta_0}(x,u\omega_0(x)-2\mathrm{Im}\,\overline{\partial}_b\phi(x))=2\pi^n\left|\det R_x^L\right|^{-1}|a_0(x,x,u)|^2\,. \tag{4.69}$$

If $\overline{\beta_0}(x,u\omega_0(x)-2\mathrm{Im}\,\overline{\partial}_b\phi(x))=0$, we get $a_0(x,x,u)=0$. If $\overline{\beta_0}(x,u\omega_0(x)-2\mathrm{Im}\,\overline{\partial}_b\phi(x))\neq0$, in view of (4.55), we know that $a_0(x,x,u)\neq0$. From this observation and (4.69), we obtain

Theorem 4.16. *For $a_0(x,y,u)$ in* (4.53)*,*

$$a_0(x,x,u)=\frac{1}{2}\pi^{-n}\left|\det R_x^L\right|\beta_0\big(x,u\omega_0(x)-2\mathrm{Im}\,\overline{\partial}_b\phi(x)\big),\ \ (x,x,u)\in D\times D\times(-M,M),$$

where $\beta_0(x,\eta)\in\mathscr{C}^\infty(T^*D)$ is as in (4.54) *and* $\det R_x^L$ *as in* (1.1).

5. Regularity of the Szegő projection Π_k

In this section, we will prove Theorem 1.1. For this purpose we first establish the spectral gap for the Kohn Laplacian $\Box^{(1)}_{b,k}$ and then Sobolev estimates for the associated Green operator and finally for Π_k.

We start with a local form of the spectral gap estimate for $(0,1)$-forms.

Lemma 5.1. *Let s be a local trivializing section of L on an open set $D\subset X$. Then, there is a constant $C>0$ independent of k such that*

$$\left\|\overline{\partial}_{b,k}u\right\|_k^2+\|\overline{\partial}^*_{b,k}u\|_k^2\geq\Big(Ck-\frac{1}{C}\Big)\left\|u\right\|_k^2,\ \ \forall u\in\Omega_0^{0,1}(D,L^k).$$

Proof. Let $u\in\Omega_0^{0,1}(D,L^k)$. Put $u=s^k\widehat{u}$, $\widehat{u}\in\Omega_0^{0,1}(D)$. In view of (3.3), we have

$$\Box^{(1)}_{b,k}u=e^{k\phi}s^k\Box^{(1)}_{s,k}(e^{-k\phi}\widehat{u}). \tag{5.1}$$

Put $\widehat{u}=\sum_{j=1}^{n-1}\widehat{u}_je_j$, where $e_1,\ldots,e_{n-1}\in T^{*0,1}X$ is as in Proposition 3.1. From (3.6), we have

$$
\begin{aligned}
&(\Box^{(1)}_{s,k}(e^{-k\phi}\widehat{u})\,|\,e^{-k\phi}\widehat{u}\,)\\
&=\sum_{j=1}^{n-1}\big\|(Z_j+kZ_j(\phi))(e^{-k\phi}\widehat{u})\big\|^2+\sum_{j,t=1}^{n-1}(\,[Z_j+kZ_j(\phi),-\overline{Z}_t+k\overline{Z}_t(\phi)](e^{-k\phi}\widehat{u}_t)\,|\,e^{-k\phi}\widehat{u}_j\,)\\
&\quad+(\,(\varepsilon(Z+kZ(\phi))+\varepsilon(Z^*+k\overline{Z}(\phi)))(e^{-k\phi}\widehat{u})\,|\,e^{-k\phi}\widehat{u}\,)+(\,fe^{-k\phi}\widehat{u}\,|\,e^{-k\phi}\widehat{u}\,).
\end{aligned}
\tag{5.2}
$$

Here we use the same notations as in Proposition 3.1. Fix $j,t=1,2,\ldots,n-1$. Put

$$
[Z_j,-\overline{Z}_t]=\sum_{s=1}^{n-1}(a^{j,t}_sZ_s-b^{j,t}_s\overline{Z}_s),\quad a^{j,t}_s,b^{j,t}_s\in\mathscr{C}^\infty(D),\quad\forall s.
$$

Recall than by [14, Lemma 4.1], for any $U,V\in T^{1,0}_pX$ and any $\mathcal{U},\mathcal{V}\in C^\infty(D,T^{1,0}X)$ that satisfy $\mathcal{U}(p)=U$, $\mathcal{V}(p)=V$, we have

$$
R^L_p(U,V)=M^\phi_p(U,V)=-\big\langle\big[\,\mathcal{U},\overline{\mathcal{V}}\,\big](p),\overline{\partial}_b\phi(p)-\partial_b\phi(p)\big\rangle+\big(\mathcal{U}\overline{\mathcal{V}}+\overline{\mathcal{V}}\mathcal{U}\big)\phi(p).
\tag{5.3}
$$

By using (5.3) we obtain

$$
\begin{aligned}
&[Z_j+kZ_j(\phi),-\overline{Z}_t+k\overline{Z}_t(\phi)]\\
&=\sum_{s=1}^{n-1}(a^{j,t}_sZ_s-b^{j,t}_s\overline{Z}_s)+k(Z_j\overline{Z}_t+\overline{Z}_tZ_j)(\phi)\\
&=\sum_{s=1}^{n-1}(a^{j,t}_s(Z_s+kZ_s(\phi))+b^{j,t}_s(-\overline{Z}_s+k\overline{Z}_s(\phi)))\\
&\quad-k\langle\,[Z_j,-\overline{Z}_t]\,,\,\overline{\partial}_b\phi-\partial_b\phi\,\rangle+k(Z_j\overline{Z}_t+\overline{Z}_tZ_j)(\phi)\\
&=\varepsilon(Z+kZ(\phi))+\varepsilon(-\overline{Z}+k\overline{Z}(\phi))+kR^L_x(\overline{Z}_t,Z_j).
\end{aligned}
\tag{5.4}
$$

From (5.4) and (5.2), we get

$$
\begin{aligned}
&\big(\Box^{(1)}_{s,k}(e^{-k\phi}\widehat{u})\,|\,e^{-k\phi}\widehat{u}\,\big)\\
&=\sum_{j=1}^{n-1}\big\|(Z_j+kZ_j(\phi))(e^{-k\phi}\widehat{u})\big\|^2+k\sum_{j,t=1}^{n-1}\big(\,R^L_x(\overline{Z}_t,Z_j)(e^{-k\phi}\widehat{u}_t)\,|\,e^{-k\phi}\widehat{u}_j\,\big)\\
&\quad+(\,(\varepsilon(Z+kZ(\phi))+\varepsilon(Z^*+k\overline{Z}(\phi)))(e^{-k\phi}\widehat{u})\,|\,e^{-k\phi}\widehat{u}\,)+(\,\widetilde{f}e^{-k\phi}\widehat{u}\,|\,e^{-k\phi}\widehat{u}\,),
\end{aligned}
\tag{5.5}
$$

where $\widetilde{f}$ is a smooth function independent of k. Since $R^L>0$, from (5.5), it is not difficult to see that

$$
\Big(\Box^{(1)}_{s,k}(e^{-k\phi}\widehat{u})\,|\,e^{-k\phi}\widehat{u}\Big)\geq\Big(\widetilde{C}k-\frac{1}{\widetilde{C}}\Big)\,\big\|e^{-k\phi}\widehat{u}\big\|^2,
\tag{5.6}
$$

where $\widetilde{C}>0$ is a constant independent of k and u. From (5.1), we can check that

$$
\big(\,\Box^{(1)}_{s,k}(e^{-k\phi}\widehat{u})\,|\,e^{-k\phi}\widehat{u}\,\big)=\big(\,\Box^{(1)}_{b,k}u\,|\,u\,\big)_k=\|\overline{\partial}_{b,k}u\|^2_k+\|\overline{\partial}^*_{b,k}\widehat{u}\|^2_k.
$$

Moreover, it is clearly that $\|u\|_k=\|e^{-k\phi}\widehat{u}\|$. From this observation and (5.6), the lemma follows. $\Box$

Ohsawa and Sibony [25] established analogues of the Nakano and Akizuki vanishing theorems for Levi flat CR manifolds. The following result can be seen as an analogue of the spectral gap and Kodaira-Serre vanishing theorem [19, Theorems 1.5.5-6].

Theorem 5.2. *There is a constant $C_0 > 0$ independent of k such that*

$$\left\|\overline{\partial}_{b,k}u\right\|_k^2 + \|\overline{\partial}^*_{b,k}u\|_k^2 \geq \left(C_0k - \frac{1}{C_0}\right)\|u\|_k^2, \quad \forall u \in \operatorname{Dom}\overline{\partial}_{b,k} \cap \operatorname{Dom}\overline{\partial}^*_{b,k} \subset L^2_{(0,1)}(X, L^k).$$

Hence, for k large, $\operatorname{Ker}\Box^{(1)}_{b,k} = \{0\}$ and $\Box^{(1)}_{b,k}$ has L^2 closed range.

From Theorem 5.2, we deduce that $\Box^{(1)}_{b,k}$ is injective for large k so we can consider the Green operator $N^{(1)}_k : L^2_{(0,1)}(X, L^k) \to \operatorname{Dom}\Box^{(1)}_{b,k}$, which is the inverse of $\Box^{(1)}_{b,k}$. We have

$$\begin{aligned} \Box^{(1)}_{b,k}N^{(1)}_k &= I \text{ on } L^2_{(0,1)}(X), \\ N^{(1)}_k\Box^{(1)}_{b,k} &= I \text{ on } \operatorname{Dom}\Box^{(1)}_{b,k}. \end{aligned} \tag{5.7}$$

Proof. We first claim that there is a constant $C_0 > 0$ independent of k such that

$$\left\|\overline{\partial}_{b,k}u\right\|_k^2 + \|\overline{\partial}^*_{b,k}u\|_k^2 \geq \left(C_0k - \frac{1}{C_0}\right)\|u\|_k^2, \quad \forall u \in \Omega^{0,1}(X, L^k). \tag{5.8}$$

Let $X = \bigcup_{j=1}^N D_j$, where $D_j \subset X$ is an open set with $L|_{D_j}$ is trivial. Take $\chi_j \in \mathscr{C}^\infty_0(D_j, [0,1])$, $j = 1, \ldots, N$, with $\sum\limits_{j=1}^N \chi_j = 1$ on X. Let $u \in \Omega^{0,1}(D, L^k)$. From Lemma 5.1, we see that for every $j = 1, 2, \ldots, N$, we can find a constant $C_j > 0$ independent of k and u such that

$$\left\|\overline{\partial}_{b,k}(\chi_ju)\right\|_k^2 + \|\overline{\partial}^*_{b,k}(\chi_ju)\|_k^2 \geq \left(C_jk - \frac{1}{C_j}\right)\|\chi_ju\|_k^2. \tag{5.9}$$

It is easy to see that

$$\begin{aligned} \left\|\overline{\partial}_{b,k}(\chi_ju)\right\|_k^2 + \|\overline{\partial}^*_{b,k}(\chi_ju)\|_k^2 &\leq \left\|\chi_j\overline{\partial}_{b,k}u\right\|_k^2 + \|\chi_j\overline{\partial}^*_{b,k}u\|_k^2 + M_j\|u\|_k^2 \\ &\leq \left\|\overline{\partial}_{b,k}u\right\|_k^2 + \|\overline{\partial}^*_{b,k}u\|_k^2 + M_j\|u\|_k^2, \end{aligned} \tag{5.10}$$

where $M_j > 0$ is a constant independent of k and u. From (5.10) and (5.9), we get

$$\begin{aligned} &N\Big(\left\|\overline{\partial}_{b,k}u\right\|_k^2 + \|\overline{\partial}^*_{b,k}u\|_k^2\Big) \\ &\geq \sum_{j=1}^N\left(\left(C_jk - \frac{1}{C_j}\right)\|\chi_ju\|_k^2 - M_j\|u\|_k^2\right) \\ &\geq \left(ck - \frac{1}{c}\right)\|u\|_k^2, \end{aligned} \tag{5.11}$$

where $c > 0$ is a constant independent of k. From (5.11), the claim (5.8) follows.

Now, let $u \in \operatorname{Dom}\overline{\partial}_{b,k} \cap \operatorname{Dom}\overline{\partial}^*_{b,k}$. From Friedrichs' Lemma (see Appendix D in [4]), we can find $u_j \in \Omega^{0,1}(X, L^k)$, $j = 1, 2, \ldots$, with $u_j \to u$ in $L^2_{(0,1)}(X, L^k)$, $\overline{\partial}_{b,k}u_j \to \overline{\partial}_{b,k}u$ in $L^2_{(0,2)}(X, L^k)$ and $\overline{\partial}^*_{b,k}u_j \to \overline{\partial}^*_{b,k}u$ in $L^2(X, L^k)$. From (5.8), we have

$$\begin{aligned} \left\|\overline{\partial}_{b,k}u\right\|_k^2 + \|\overline{\partial}^*_{b,k}u\|_k^2 &= \lim_{j\to\infty}\Big(\left\|\overline{\partial}_{b,k}u_j\right\|_k^2 + \|\overline{\partial}^*_{b,k}u_j\|_k^2\Big) \\ &\geq \left(C_0k - \frac{1}{C_0}\right)\lim_{j\to\infty}\|u_j\|_k^2 = \left(C_0k - \frac{1}{C_0}\right)\|u\|_k^2. \end{aligned}$$

The theorem follows. ☐

We pause and introduce some notations. Let s be a local trivializing section of L on an open set $D \subset X$, $|s|^2_h = e^{-2\phi}$. Let $u \in \Omega^{0,q}_0(D, L^k)$. On D, we write $u = s^k \widetilde{u}$, $\widetilde{u} \in \Omega^{0,q}_0(D)$. For every $m \in \mathbb{N}_0$, define

$$\|u\|^2_{m,k} := \sum_{|\alpha| \leq m, \alpha \in \mathbb{N}^{2n-1}_0} \int \left|\partial^\alpha_x(\widetilde{u}e^{-k\phi})\right|^2 dv_X.$$

By using a partition of unity, we can define $\|u\|^2_{m,k}$ for all $u \in \Omega^{0,q}(X, L^k)$ in the standard way. We call $\|\cdot\|_{m,k}$ the Sobolev norm of order m with respect to h^k. We will need the following.

Proposition 5.3 ([25, Proposition 1]). *For every $m \in \mathbb{N}_0$ there is $N_m > 0$ such that for every $k \geq N_m$,*

$$\left\|\overline{\partial}^*_{b,k}u\right\|_{m,k} \leq k^{M(m)} \left\|\Box^{(1)}_{b,k}u\right\|_{m,k}, \quad \forall u \in \Omega^{0,1}(X, L^k), \tag{5.12}$$

where $M(m) > 0$ is a constant independent of k and u.

Theorem 5.4. *For every $m \in \mathbb{N}$, there is $N_m > 0$ such that for every $k \geq N_m$,*

$$\overline{\partial}^*_{b,k}N^{(1)}_k : \Omega^{0,1}(X, L^k) \to H^m(X, L^k)$$

and

$$\|\overline{\partial}^*_{b,k}N^{(1)}_k u\|_{m,k} \leq k^{M(m)} \|u\|_{m,k}, \quad \forall u \in \Omega^{0,1}(X, L^k),$$

where $M(m) > 0$ is a constant independent of k and u.

Proof. The theorem essentially follows from Proposition 5.3 and the elliptic regularization method introduced by Kohn-Nirenberg [4, p. 102], [17, p. 449]. Namely, for every $\varepsilon > 0$, consider the operator $\Box^{(1)}_{\varepsilon,k} := \Box^{(1)}_{b,k} + \varepsilon T^*T$, where T is defined in (2.10) and T^* is its formal adjoint with respect to $(\,\cdot\,|\,\cdot\,)_k$. Fix $m \in \mathbb{N}$. From Theorem 5.2 and Proposition 5.3, there is a $N_m > 0$ such that for every $k \geq N_m$,

$$\begin{aligned} &\|u\|^2_k \leq (\,\Box^{(1)}_{b,k}u\,|\,u\,)_k, \quad \forall u \in \Omega^{0,1}(X, L^k), \\ &\|u\|_{\ell,k} \leq k^{M(m)}\|\Box^{(1)}_{b,k}u\|_{\ell,k}, \quad \forall u \in \Omega^{0,1}(X, L^k), \quad \forall \ell \in \mathbb{N}_0, \quad \ell \leq m, \end{aligned} \tag{5.13}$$

where $M(m) > 0$ is a constant independent of k and u.

Take $g \in \Omega^{0,1}(X, L^k)$ and put $N^{(1)}_k g = v$. We have $\Box^{(1)}_{b,k}v = g$. From (5.13), it is easy to see that for every $k \geq N_m$ and every $\varepsilon > 0$, $\Box^{(1)}_{\varepsilon,k}$ is injective and has range $L^2_{(0,1)}(X, L^k)$. Now, we assume that $k \geq N_m$. For every $\varepsilon > 0$, we can find $v_\varepsilon \in \Omega^{0,1}(X, L^k)$ such that $\Box^{(1)}_{\varepsilon,k}v_\varepsilon = g$. Moreover, from (5.13) and the proof of Proposition 5.3 (see also [25, Proposition 1]), it is straightforward to see that for every $\varepsilon > 0$,

$$\begin{aligned} &\|v_\varepsilon\|_k \leq \|g\|_k, \quad \|\overline{\partial}_{b,k}v_\varepsilon\|_k \leq \|g\|_k, \\ &\|\overline{\partial}^*_{b,k}v_\varepsilon\|_{\ell,k} \leq k^{M(m)} \|g\|_{\ell,k}, \quad \forall \ell \in \mathbb{N}_0, \quad \ell \leq m. \end{aligned} \tag{5.14}$$

From (5.14), we can find $\varepsilon_j \searrow 0$ such that $v_{\varepsilon_j} \to \widetilde{v}$ in $L^2_{(0,1)}(X, L^k)$ as $j \to \infty$, $\overline{\partial}_{b,k}v_{\varepsilon_j} \to \overline{\partial}_{b,k}\widetilde{v}$ in $L^2_{(0,2)}(X, L^k)$, $\overline{\partial}^*_{b,k}v_{\varepsilon_j} \to \overline{\partial}^*_{b,k}\widetilde{v}$ in $H^\ell(X, L^k)$, $\forall \ell \in \mathbb{N}_0$, $\ell \leq m$, and $\Box^{(1)}_{b,k}\widetilde{v} = g$ in the sense of distributions. Since $\overline{\partial}_{b,k}\widetilde{v} \in L^2_{(0,2)}(X, L^k)$, $\overline{\partial}^*_{b,k}\widetilde{v} \in H^1(X, L^k)$, we have $\widetilde{v} \in \operatorname{Dom}\overline{\partial}_{b,k} \cap \operatorname{Dom}\overline{\partial}^*_{b,k}$, $\overline{\partial}^*_{b,k}\widetilde{v} \in \operatorname{Dom}\overline{\partial}_{b,k}$. Note that $\overline{\partial}^*_{b,k}\overline{\partial}_{b,k}\widetilde{v} = g - \overline{\partial}_{b,k}\overline{\partial}^*_{b,k}\widetilde{v} \in L^2_{(0,1)}(X, L^k)$. From this observation,

we can check that $\overline{\partial}_{b,k}\widetilde{v} \in \operatorname{Dom}\overline{\partial}^*_{b,k}$. Thus, $\widetilde{v} \in \operatorname{Dom}\Box^{(1)}_{b,k}$. Since $\Box^{(1)}_{b,k}\widetilde{v} = g = \Box^{(1)}_{b,k}v$ and $\Box^{(1)}_{b,k}$ is injective, we conclude that $v = \widetilde{v}$. Thus, $\overline{\partial}^*_{b,k}N^{(1)}_k g = \overline{\partial}^*_{b,k}v \in H^m(X, L^k)$ and $\|\overline{\partial}^*_{b,k}N^{(1)}_k g\|_{m,k} \leq k^{M(m)}\|g\|_{m,k}$. The theorem follows. $\square$

Theorem 5.5. *With the notations above, for every $m \in \mathbb{N}$, $m \geq 2$, there is a $N_m > 0$ such that for every $k \geq N_m$,*

$$\Pi_k = I - \overline{\partial}^*_{b,k}N^{(1)}_k\overline{\partial}_{b,k} \text{ on } \mathscr{C}^\infty(X, L^k), \tag{5.15}$$

$$\Pi_k : \mathscr{C}^\infty(X, L^k) \to H^m(X, L^k) \tag{5.16}$$

and

$$\|(I - \Pi_k)u\|_{m,k} \leq k^{M(m)}\left\|\overline{\partial}_{b,k}u\right\|_{m,k}, \quad \forall u \in \mathscr{C}^\infty(X, L^k), \tag{5.17}$$

where $M(m) > 0$ is a constant independent of k and u.

Proof. Fix $m \in \mathbb{N}$, $m \geq 2$ and let $N_m > 0$ be as in Theorem 5.5. We assume that $k \geq N_m$. Let $g \in \mathscr{C}^\infty(X, L^k)$. From Theorem 5.4, we know that $\overline{\partial}^*_{b,k}N^{(1)}_k\overline{\partial}_{b,k}g \in H^m(X, L^k)$. Since $m \geq 2$, it is clearly that $\overline{\partial}^*_{b,k}N^{(1)}_k\overline{\partial}_{b,k}g \in \operatorname{Dom}\Box^{(0)}_{b,k}$. Moreover, it is easy to check that

$$\overline{\partial}^*_{b,k}N^{(1)}_k\overline{\partial}_{b,k}g \perp \operatorname{Ker}\overline{\partial}_{b,k} = \operatorname{Ker}\Box^{(0)}_{b,k}. \tag{5.18}$$

We claim that

$$g - \overline{\partial}^*_{b,k}N^{(1)}_k\overline{\partial}_{b,k}g \in \operatorname{Ker}\Box^{(0)}_{b,k}. \tag{5.19}$$

Let $f \in \mathscr{C}^\infty(X, L^k)$. We have

$$\begin{aligned}
(\, g - \overline{\partial}^*_{b,k}N^{(1)}_k\overline{\partial}_{b,k}g \,|\, \Box^{(0)}_{b,k}f \,)_k &= (\, \Box^{(0)}_{b,k}g \,|\, f \,)_k - (\, \overline{\partial}^*_{b,k}N^{(1)}_k\overline{\partial}_{b,k}g \,|\, \Box^{(0)}_{b,k}f \,)_k \\
&= (\, \Box^{(0)}_{b,k}g \,|\, f \,)_k - (\, \overline{\partial}_{b,k}g \,|\, N^{(1)}_k\overline{\partial}_{b,k}\Box^{(0)}_{b,k}f \,)_k \\
&= (\, \Box^{(0)}_{b,k}g \,|\, f \,)_k - (\, \overline{\partial}_{b,k}g \,|\, N^{(1)}_k\Box^{(1)}_{b,k}\overline{\partial}_{b,k}f \,)_k \\
&= (\, \Box^{(0)}_{b,k}g \,|\, f \,)_k - (\, \overline{\partial}_{b,k}g \,|\, \overline{\partial}_{b,k}f \,)_k = 0.
\end{aligned}$$

The claim (5.19) follows. From (5.18) and (5.19), we get (5.15).

From (5.15) and Theorem 5.4, we get (5.16) and (5.17). $\square$

From Theorem 5.5 and Sobolev embedding theorem, we get Theorem 1.1.

6. Asymptotic expansion of the Szegő kernel

In this section, we will prove Theorem 1.2 and Theorem 1.3. Let s be a local trivializing section of L on an open set $D \subset X$ and let $\Pi_{k,s}$ be the localized operator of Π_k (see (1.4)). Let $\mathcal{S}_k$ and $\mathcal{G}_k$ be as in Theorem 4.12. From the constructions of $\mathcal{G}_k$ and $\mathcal{S}_k$, it is straightforward to see that we can find $\widetilde{\mathcal{G}_k}; H^s_{\mathrm{comp}}(D) \to H^{s+1}_{\mathrm{loc}}(D)$, $\widetilde{\mathcal{S}}_k : H^s_{\mathrm{comp}}(D) \to H^s_{\mathrm{loc}}(D)$, for every $s \in \mathbb{Z}$, such that $\widetilde{\mathcal{G}_k}$ and $\widetilde{\mathcal{S}_k}$ are properly supported on D,

$$\begin{aligned}
&\widetilde{\mathcal{S}}_k - \mathcal{S}_k = O(k^{-\infty}) : H^s_{\mathrm{comp}}(D) \to H^s_{\mathrm{loc}}(D), \ \ \forall s \in \mathbb{Z}, \\
&\widetilde{\mathcal{G}_k} - \mathcal{G}_k = O(k^{-\infty}) : H^s_{\mathrm{comp}}(D) \to H^{s+1}_{\mathrm{loc}}(D), \ \ \forall s \in \mathbb{Z},
\end{aligned} \tag{6.1}$$

and

$$\widetilde{\chi}\widetilde{\mathcal{S}}_k\chi = O(k^{-\infty}) : H^s_{\mathrm{comp}}(D) \to H^s_{\mathrm{loc}}(D), \ \ \forall s \in \mathbb{Z}, \tag{6.2}$$

for every $\widetilde{\chi}, \chi \in C_0^\infty(D)$ with $\operatorname{Supp}\widetilde{\chi}\bigcap \operatorname{Supp}\chi = \emptyset$, and

$$\Box^{(0)}_{s,k}\widetilde{\mathcal{G}_k} + \widetilde{\mathcal{S}_k} = I + R_k \text{ on } D, \tag{6.3}$$

where R_k is properly supported on D and

$$R_k = O(k^{-\infty}) : H^s_{\mathrm{loc}}(D) \to H^{s-1}_{\mathrm{loc}}(D),\ \ \forall s \in \mathbb{Z}. \tag{6.4}$$

From (6.3), it is easy to see that

$$\Pi_{k,s} + \Pi_{k,s}R_k = \Pi_{k,s}\widetilde{\mathcal{S}_k} \text{ on } D. \tag{6.5}$$

Theorem 6.1. *With the notations above, for every $\ell \in \mathbb{N}_0$, there is a $N_\ell > 0$ such that for every $k \geq N_\ell$, $\widetilde{\chi}\Pi_k\chi = O(k^{-\infty}) : \mathscr{C}^\infty(X, L^k) \to \mathscr{C}^\ell(X, L^k)$, for every $\chi \in \mathscr{C}^\infty_0(D)$, $\widetilde{\chi} \in \mathscr{C}^\infty(X)$ with $\operatorname{Supp}\widetilde{\chi}\bigcap\operatorname{Supp}\chi = \emptyset$, and*

$$\Pi_{k,s}(x,y) - \mathcal{S}_k(x,y) = O(k^{-\infty}) : \mathscr{C}^\infty_0(D) \to \mathscr{C}^\ell(D). \tag{6.6}$$

Proof. Fix $\ell \in \mathbb{N}_0$. From Theorem 5.5, there is a $N_\ell > 0$ such that for every $k \geq N_\ell$,

$$\begin{aligned} &\Pi_k = I - \overline{\partial}^*_{b,k}N^{(1)}_k\overline{\partial}_{b,k} \text{ on } \mathscr{C}^\infty(X, L^k),\\ &\Pi_k : \mathscr{C}^\infty(X, L^k) \to H^{\ell+n}(X, L^k),\\ &\|(I-\Pi_k)u\|_{n+\ell,h^k} \leq k^{M(\ell)}\left\|\overline{\partial}_{b,k}u\right\|_{n+\ell,h^k},\ \ \forall u \in \mathscr{C}^\infty(X, L^k),\end{aligned} \tag{6.7}$$

where $M(\ell) > 0$ is a constant independent of k and u. Now, we assume that $k \geq N_\ell$. By the Sobolev embedding theorem we have $H^{\ell+n}(X, L^k) \subset \mathscr{C}^\ell(X, L^k)$.

Fix $N_1 > 0$ and let $u \in \mathscr{C}^\infty_0(D)$. Consider

$$v = U_{k,s}\widetilde{\mathcal{S}_k}u - \Pi_k(U_{k,s}\widetilde{\mathcal{S}_k}u) = (I-\Pi_k)(U_{k,s}\widetilde{\mathcal{S}_k}u). \tag{6.8}$$

From (6.5), we have

$$\begin{aligned} &v = U_{k,s}(\widetilde{\mathcal{S}_k} - \Pi_{k,s}\widetilde{\mathcal{S}_k})u \text{ on } D,\\ &v = U_{k,s}(\widetilde{\mathcal{S}_k}u) - \Pi_k(U_{k,s}(I+R_k)u) \text{ on } X.\end{aligned} \tag{6.9}$$

From (6.7) and (6.8), we obtain

$$\left\|(I-\Pi_k)(U_{k,s}\widetilde{\mathcal{S}_k}u)\right\|_{n+\ell,h^k} \leq k^{M(\ell)}\left\|\overline{\partial}_{b,k}(U_{k,s}\widetilde{\mathcal{S}_k}u)\right\|_{n+\ell,h^k}. \tag{6.10}$$

Note that $\overline{\partial}_{s,k}\widetilde{\mathcal{S}_k} = O(k^{-\infty}) : H^s_{\mathrm{comp}}(D) \to H^{s-1}_{\mathrm{loc}}(D)$ for all $s \in \mathbb{Z}$. From this observation, (6.10) and the second formula of (6.9) we conclude that

$$U_{k,s}\widetilde{\mathcal{S}_k} - \Pi_kU_{k,s} - \Pi_kU_{k,s}R_k = O(k^{-\infty}) : \mathscr{C}^\infty_0(D) \to \mathscr{C}^\ell(X, L^k). \tag{6.11}$$

From (6.4) and (6.7), it is easy to see that

$$\Pi_kU_{k,s}R_k = O(k^{-\infty}) : \mathscr{C}^\infty_0(D) \to \mathscr{C}^\ell(X, L^k). \tag{6.12}$$

From (6.11) and (6.12), we conclude that

$$U_{k,s}\widetilde{\mathcal{S}_k} - \Pi_kU_{k,s} = O(k^{-\infty}) : \mathscr{C}^\infty_0(D) \to \mathscr{C}^\ell(X, L^k). \tag{6.13}$$

From (6.13) and (6.1), (6.6) follows.

Finally, from (6.13), (6.2) and noting that $\widetilde{\mathcal{S}_k}$ is properly supported on D, we deduce that $\widetilde{\chi}\Pi_k\chi = O(k^{-\infty}) : \mathscr{C}^\infty(X, L^k) \to \mathscr{C}^\ell(X, L^k)$, for every $\chi \in \mathscr{C}^\infty_0(D)$, $\widetilde{\chi} \in \mathscr{C}^\infty(X)$ with $\operatorname{Supp}\widetilde{\chi} \cap \operatorname{Supp}\chi = \emptyset$. $\square$

Proof of Theorem 1.3. Let $\mathcal{A}_k$ be as in Theorem 1.3. It is not difficult to see that for every $s\in\mathbb{Z}$ and $N\in\mathbb{N}$, there is a $n(N,s)>0$ indpendent of k such that

$$\mathcal{A}_k=O(k^{n(N,s)}):H^s_{\rm comp}(D)\to\mathscr{C}^N_0(D). \tag{6.14}$$

From (6.14), (6.6) and since $\mathcal{A}_k:H^s_{\rm comp}(D)\to\mathscr{C}^\infty_0(D)$ for every $s\in\mathbb{Z}$, we conclude that

$$\Pi_{k,s}\mathcal{A}_k\equiv\mathcal{S}_k\mathcal{A}_k\mod O(k^{-\infty}). \tag{6.15}$$

From (6.15) and Theorem 4.14, Theorem 1.3 follows. □

7. Kodaira Embedding theorem for Levi-flat CR manifolds

In this section, we will prove Theorem 1.4. Let s be a local trivializing section of L on an open set $D\subset X$. Fix $p\in D$ and let $x=(x_1,\ldots,x_{2n-1})$, $z_j=x_{2j-1}+ix_{2j}$, $j=1,\ldots,n-1$, be local coordinates of X defined in some small neighbourhood of p such that (4.31) hold. We may assume that the local coordinates x defined on D. We write $x'=(x_1,\ldots,x_{2n-2})$. Let $M>1$ be a large constant so that

$$\left|-2{\rm Im}\,\overline\partial_b\phi(x)+u\omega_0(x)\right|^2\leq\frac{M^2}{8},\ \ \forall x\in D,|u|\leq1. \tag{7.1}$$

Take $\tau\in\mathscr{C}^\infty_0(\mathbb{R},[0,1])$ with $\tau=1$ on $[\frac14,\frac12]$, $\tau=0$ on $]-\infty,0]\bigcup[1,\infty[$ and take $\chi\in\mathscr{C}^\infty_0(\mathbb{R},[0,1])$ with $\chi=1$ on $[-\frac12,\frac12]$, $\chi=0$ on $]-\infty,-1]\bigcup[1,\infty[$ and $\chi(t)=\chi(-t)$, for every $t\in\mathbb{R}$. Fix $0<\delta<1$. Put

$$\alpha_\delta(x,\eta,k):=\tau\Big(\frac{\langle\,\eta\,|\,\omega_0(x)\,\rangle}{\delta}\Big)\chi\Big(\frac{4\,|\eta|^2}{M^2}\Big)\in S^0_{\rm cl}(1,T^*D) \tag{7.2}$$

and let $\mathcal{A}_{k,\delta}$ be a properly supported classical semi-classical pseudodifferential operator on D with

$$\mathcal{A}_{k,\delta}(x,y)\equiv\frac{k^{2n-1}}{(2\pi)^{2n-1}}\int e^{ik\langle x-y,\eta\rangle}\alpha_\delta(x,\eta,k)d\eta\mod O(k^{-\infty}).$$

Fix $\ell\in\mathbb{N}$, $\ell\geq2$. In view of Theorem 1.3, we see that there is a $N_\ell>0$ such that for every $k\geq N_\ell$, $\Pi_{k,s}\mathcal{A}_{k,\delta}(x,y)\in\mathscr{C}^\ell(D\times D)$ and

$$(\Pi_{k,s}\mathcal{A}_{k,\delta})(x,y)\equiv\int e^{ik\psi(x,y,u)}a_\delta(x,y,u,k)du\mod O(k^{-\infty})\ \text{ in }\mathscr{C}^\ell(D\times D), \tag{7.3}$$

where

$$\begin{aligned}&a_\delta(x,y,u,k)\in\mathscr{C}^\infty_0(D\times D\times(-M,M))\cap S^n_{{\rm loc},{\rm cl}}(1;D\times D\times(-M,M)),\\&a_\delta(x,y,u,k)\sim\sum^\infty_{j=0}a_{j,\delta}(x,y,u)k^{n-j}\text{ in }S^n_{\rm loc}(1;D\times D\times(-M,M)),\\&a_{j,\delta}(x,y,u)\in\mathscr{C}^\infty_0(D\times D\times(-M,M)),\ \ j=0,1,2,\ldots.\end{aligned} \tag{7.4}$$

From (1.11), (7.1) and (7.3), we get

$$a_{0,\delta}(x,x,u)=\frac12\pi^{-n}\left|\det R^L_x\right|\tau\Big(\frac{u}{\delta}\Big),\ \ \forall(x,x,u)\in D\times D\times(-M,M). \tag{7.5}$$

From now on, we assume that $k\geq N_\ell$. Let

$$u_{k,\delta,p}:=\Pi_kU_{k,s}\mathcal{A}_{k,\delta}\Big(e^{k(\sum^{n-1}_{l=1}(\alpha_lw_l-\overline\alpha_l\overline w_l)+iuy_{2n-1}+\frac12\sum^{n-1}_{j=1}\lambda_j|w_j|^2)}\chi(ky_{2n-1})\chi(\sqrt{k}y_1)\ldots\chi(\sqrt{k}y_{2n-2})\Big), \tag{7.6}$$

where $w_j = y_{2j-1} + iy_{2j}$ and $\alpha_j \in \mathbb{C}$, $j = 1, \ldots, n-1$, are as in (4.31). Then, $u_{k,\delta,p}$ is a global $\mathscr{C}^\ell$ CR section. On D, we write $u_{k,\delta,p} = U_{k,s}\widetilde{u}_{k,\delta,p}$, $\widetilde{u}_{k,\delta,p} \in C^\ell(D)$. Then,

$$|u_{k,\delta,p}(x)|_{h^k} = |\widetilde{u}_{k,\delta,p}(x)|\,,\ \ x \in D.$$

Put $\psi_0(x,y,u) := \psi(x,y,u) - i\sum_{j=1}^{n-1}(\alpha_j w_j - \overline{\alpha}_j\overline{w}_j) + uy_{2n-1} - \frac{i}{2}\sum_{j=1}^{n-1}\lambda_j\,|w_j|^2$. From (7.3), we can check that we have $\operatorname{mod} O(k^{-\infty})$ in $\mathscr{C}^\ell(D)$,

$$\begin{aligned}\widetilde{u}_{k,\delta,p}(x) &\equiv \int e^{ik\psi_0(x,y,u)}a_\delta(x,y,u,k)\chi(ky_{2n-1})\chi(\sqrt{k}y_1)\cdots\chi(\sqrt{k}y_{2n-2})\\ &\equiv \int e^{ik\psi_0(x,F_k^*y,u)}k^{-n}a_\delta(x,F_k^*y,u,k)\chi(y_{2n-1})\chi(y_1)\cdots\chi(y_{2n-2})dudy,\end{aligned} \tag{7.7}$$

where $F_k^*y := \Big(\dfrac{y_1}{\sqrt{k}}, \dfrac{y_2}{\sqrt{k}}, \cdots, \dfrac{y_{2n-2}}{\sqrt{k}}, \dfrac{y_{2n-1}}{k}\Big)$. Put

$$\widehat{u}_{k,\delta,p} := \exp\Big(-k\sum_{j=1}^{n-1}(\alpha_j z_j - \overline{\alpha}_j\overline{z}_j)\Big)\widetilde{u}_{k,\delta,p} \in \mathscr{C}^\ell(D). \tag{7.8}$$

Lemma 7.1. *With the notations above, there is a $k_0 > 0$ such for all $k \geq k_0$ and $p \in X$,*

$$\begin{aligned}&\frac{1}{8}\delta c_p \leq |\widehat{u}_{k,\delta,p})(p)| \leq 2\delta c_p,\\ &\frac{1}{32}\delta^2 c_p \leq \left|\frac{1}{k}\frac{\partial\widehat{u}_{k,\delta,p}}{\partial x_{2n-1}}(p)\right| \leq 2\delta^2 c_p,\\ &\left|\frac{1}{k}\frac{\partial\widehat{u}_{k,\delta,p}}{\partial x_j}(p)\right| \leq \delta^4,\ \ j = 1,2,\ldots,2n-2,\end{aligned} \tag{7.9}$$

where $c_p = \frac{1}{2}\pi^{-n}\left|\det R_p^L\right|\int\chi(y_{2n-1})\chi(y_1)\cdots\chi(y_{2n-2})dy$.

Proof. From (7.7), (7.5), (4.33) and note that $\psi_0(0,0,u) = 0$, $\forall u \in \mathbb{R}$, we can check that

$$\begin{aligned}&\lim_{k\to\infty}|\widehat{u}_{k,\delta,p}(p)| = \frac{1}{2}\pi^{-n}\left|\det R_p^L\right|\int\tau(\frac{u}{\delta})\chi(y_{2n-1})\chi(y_1)\cdots\chi(y_{2n-2})dydu,\\ &\lim_{k\to\infty}\left|\frac{1}{k}\frac{\partial\widehat{u}_{k,\delta,p}}{\partial x_{2n-1}}(p)\right| = \frac{1}{2}\pi^{-n}\left|\det R_p^L\right|\int u\tau(\frac{u}{\delta})\chi(y_{2n-1})\chi(y_1)\cdots\chi(y_{2n-2})dydu,\\ &\lim_{k\to\infty}\left|\frac{1}{k}\frac{\partial\widehat{u}_{k,\delta,p}}{\partial x_j}(p)\right| = 0,\ \ j = 1,2,\ldots,2n-2.\end{aligned}$$

Since $\frac{\delta}{4} \leq \int\tau(\frac{u}{\delta})du \leq \delta$ and $\frac{\delta^2}{16} \leq \int u\tau(\frac{u}{\delta})du \leq \delta^2$, there is $k_0 > 0$ such that for every $k \geq k_0$, (7.9) hold. Since X is compact, k_0 can be taken to be independent of the point p. ☐

For every $j = 1,2,\ldots,n-1$, let

$$\begin{aligned}u^j_{k,\delta,p} :&= \Pi_k U_{k,s}\mathcal{A}_{k,\delta}\Big(e^{k(\sum_{l=1}^{n-1}(\alpha_l w_l - \overline{\alpha}_l\overline{w}_j) + iuy_{2n-1} + \frac{1}{2}\sum_{j=1}^{n-1}\lambda_j|w_j|^2)}\sqrt{k}(y_{2j-1} + iy_{2j})\\ &\quad\times\chi(ky_{2n-1})\chi(\sqrt{k}y_1)\ldots\chi(\sqrt{k}y_{2n-2})\Big).\end{aligned} \tag{7.10}$$

Then, $u^j_{k,\delta,p}$ is a global $\mathscr{C}^\ell$ CR section. On D, we write $u^j_{k,\delta,p} = U_{k,s}\widetilde{u}^j_{k,\delta,p}$, $\widetilde{u}^j_{k,\delta,p} \in \mathscr{C}^\ell(D)$. From (7.3), we can check that

$$\widetilde{u}^j_{k,\delta,p}(x) \equiv \int e^{ik\psi_0(x,F_k^*y,u)}k^{-n}a_\delta(x,F_k^*y,u,k)(y_{2j-1} + iy_{2j})\chi(y_{2n-1})\chi(y_1)\cdots\chi(y_{2n-2})dudy, \tag{7.11}$$

$\bmod O(k^{-\infty})$ in $\mathscr{C}^\ell(D)$. Put

$$\widehat{u}^j_{k,\delta,p} := \exp\Big(-k\sum_{l=1}^{n-1}(\alpha_l z_l - \overline{\alpha}_l\overline{z}_l)\Big)\widetilde{u}^j_{k,\delta,p} \in \mathscr{C}^\ell(D), \quad j=1,2,\ldots,n-1. \tag{7.12}$$

Lemma 7.2. *With the notations above, there exists $k_0>0$ such that for all $p\in X$ and $k\geq k_0$,*

$$\begin{aligned}
&\big|\widehat{u}^j_{k,\delta,p}(p)\big|\leq\delta^4,\quad \left|\frac{1}{k}\frac{\partial\widehat{u}^j_{k,\delta,p}}{\partial x_{2n-1}}(p)\right|\leq\delta^4,\quad j=1,2,\ldots,n-1,\\
&\left|\frac{1}{k}\frac{\partial\widehat{u}^j_{k,\delta,p}}{\partial\overline{z}_s}(p)\right|\leq\delta^4,\quad j,s=1,2,\ldots,n-1,\\
&\left|\frac{1}{k}\frac{\partial\widehat{u}^j_{k,\delta,p}}{\partial z_s}(p)\right|\leq\delta^4,\quad j,s=1,2,\ldots,n-1,\quad j\neq s,\\
&\left|\frac{1}{k}\frac{\partial\widehat{u}^j_{k,\delta,p}}{\partial z_j}(p)\right|\geq\frac{1}{8}\delta\lambda_j d_p,\quad j=1,2,\ldots,n-1,
\end{aligned}\tag{7.13}$$

where $\{\lambda_j\}_{j=1}^{n-1}$ are the eigenvalues of R^L_p with respect to $\langle\,\cdot\,|\,\cdot\,\rangle$ and

$$d_p=\frac{1}{2}\pi^{-n}\left|\det R^L_p\right|\int|y_1+iy_2|^2\chi(y_{2n-1})\chi(y_1)\cdots\chi(y_{2n-2})dy.$$

Proof. From (7.11), (7.5), (4.33) and observing that $\psi_0(0,0,u)=0$ for all $u\in\mathbb{R}$, it is straightforward to check that for every $j,s,t=1,\ldots,n-1$, $s\neq j$,

$$\lim_{k\to\infty}\left|\frac{1}{k}\frac{\partial\widehat{u}^j_{k,\delta,p}}{\partial z_j}(p)\right|=\frac{1}{2}\pi^{-n}\left|\det R^L_p\right|\lambda_j\int\tau\Big(\frac{u}{\delta}\Big)\left|y_{2j-1}+iy_{2j}\right|^2\chi(y_{2n-1})\chi(y_1)\cdots\chi(y_{2n-2})dydu,$$

$$\lim_{k\to\infty}\big|\widehat{u}^j_{k,\delta,p}(p)\big|=\lim_{k\to\infty}\left|\frac{1}{k}\frac{\partial\widehat{u}^j_{k,\delta}}{\partial x_{2n-1}}(p)\right|=\lim_{k\to\infty}\left|\frac{1}{k}\frac{\partial\widehat{u}^j_{k,\delta,p}}{\partial z_s}(p)\right|=\lim_{k\to\infty}\left|\frac{1}{k}\frac{\partial\widehat{u}^j_{k,\delta}}{\partial\overline{z}_t}(p)\right|=0.$$

Since $\frac{\delta}{4}\leq\int\tau(\frac{u}{\delta})du\leq\delta$, there is a constant $k_0>0$ such that (7.13) holds for every $k\geq k_0$. Since X is compact, k_0 can be taken to be independent of the point p. The lemma follows. □

Consider the $\mathscr{C}^\ell$ map

$$\Phi_{k,\delta,p}:D\to\mathbb{C}^n,\quad x\longmapsto\Big(\frac{\widetilde{u}_{k,\delta,p}}{\widetilde{u}_{k,\delta^2,p}}(x),\frac{\widetilde{u}^1_{k,\delta,p}}{\widetilde{u}_{k,\delta^2,p}}(x),\ldots,\frac{\widetilde{u}^{n-1}_{k,\delta,p}}{\widetilde{u}_{k,\delta^2,p}}(x)\Big)\in\mathbb{C}^n. \tag{7.14}$$

The following Lemma is a consequence of (7.13), (7.9) together with a straightforward computation and therefore we omit the details.

Lemma 7.3. *With the notations above, there are $k_0>0$ and $0<\delta_0<1$ such that for all $k\geq k_0$, $0<\delta\leq\delta_0$ and $p\in X$, the differential of $\Phi_{k,\delta,p}$ is injective at p.*

Let $\operatorname{dist}(\cdot,\cdot)$ denote the Riemannian distance on X and for $x\in X$ and $r>0$, put $B(x,r):=\{y\in X;\operatorname{dist}(x,y)<r\}$. From now on, we fix $k>k_0$ and $0<\delta<\delta_0$, where $k_0>0$ and $0<\delta_0<1$ are as in Lemma 7.3. Since X is compact there exists $r_k>0$ such that for every $x_0\in X$, $\widetilde{u}_{k,\delta^2,x_0}(x)\neq0$ for every $x\in B(x_0,2r_k)$ and the maps Φ_{k,δ,x_0} and $d\Phi_{k,\delta,x_0}$ are injective on $B(x_0,2r_k)$. We can find $x_1,x_2,\ldots,x_{d_k}\in X$ such that

$$X=B(x_1,r_k)\cup B(x_2,r_k)\cup\cdots\cup B(x_{d_k},r_k). \tag{7.15}$$

For every $j = 1, 2, \ldots, d_k$, let $u_{k,\delta^2,x_j}, u_{k,\delta,x_j}, u^1_{k,\delta,x_j}, \ldots, u^{n-1}_{k,\delta,x_j} \in \mathscr{C}^\ell(X, L^k)$ be as in (7.6) and (7.10). Consider the map:

$$\begin{aligned} &\Phi_{k,\delta} : X \to \mathbb{CP}^{(n+1)d_k-1}, \\ &x \longmapsto \Big[u_{k,\delta^2,x_1}, u_{k,\delta,x_1}, u^1_{k,\delta,x_1}, \ldots, u^{n-1}_{k,\delta,x_1}, \ldots, u_{k,\delta^2,x_{d_k}}, u_{k,\delta,x_{d_k}}, u^1_{k,\delta,x_{d_k}}, \ldots, u^{n-1}_{k,\delta,x_{d_k}}\Big](x). \end{aligned} \tag{7.16}$$

Let $q \in X$. Then, $q \in B(x_j, r_k)$ for some $j = 1, 2, \ldots, d_k$. From the discussion before (7.15), we see that $u_{k,\delta^2,x_j}(q) \neq 0$. Thus, $\Phi_{k,\delta}$ is well-defined as a $\mathscr{C}^\ell$ map.

Theorem 7.4. *With the notations above, the differential of $\Phi_{k,\delta}$ is injective at every $x \in X$ and for every $x_0, y_0 \in X$ with $\operatorname{dist}(x_0, y_0) \leq \frac{r_k}{2}$, we have $\Phi_{k,\delta}(x_0) \neq \Phi_{k,\delta}(y_0)$.*

Proof. Let $q \in X$. Assume that $q \in B(x_1, r_k)$. Then, $u_{k,\delta^2,x_1}(q) \neq 0$. On $B(x_1, r_k)$, consider the map:

$$\begin{aligned} &\Psi : B(x_1, r_k) \to \mathbb{C}^{(n+1)d_k-1}, \\ &\Psi(x) = \Big(\frac{u_{k,\delta,x_1}}{u_{k,\delta^2,x_1}}, \frac{u^1_{k,\delta,x_1}}{u_{k,\delta^2,x_1}}, \ldots, \frac{u^{n-1}_{k,\delta,x_1}}{u_{k,\delta^2,x_1}}, \ldots, \frac{u_{k,\delta^2,x_{d_k}}}{u_{k,\delta^2,x_1}}, \frac{u_{k,\delta,x_{d_k}}}{u_{k,\delta^2,x_1}}, \frac{u^1_{k,\delta,x_{d_k}}}{u_{k,\delta^2,x_1}}, \ldots, \frac{u^{n-1}_{k,\delta,x_{d_k}}}{u_{k,\delta^2,x_1}}\Big)(x). \end{aligned} \tag{7.17}$$

From the discussion before (7.15), we see that $d\Phi_{k,\delta,x_1}$ is injective on $B(x_1, 2r_k)$. Thus, $d\Psi$ is injective at q and hence $d\Phi_{k,\delta}$ is injective at q.

Let $x_0, y_0 \in X$ with $\operatorname{dist}(x_0, y_0) \leq \frac{r_k}{2}$. We may assume that $x_0 \in B(x_1, r_k)$. Thus, $x_0, y_0 \in B(x_1, 2r_k)$. From the discussion before (7.15), we see that Φ_{k,δ,x_1} is injective on $B(x_1, 2r_k)$. Hence,

$$\Phi_{k,\delta,x_1}(x_0) \neq \Phi_{k,\delta,x_1}(y_0). \tag{7.18}$$

From the definition of Φ_{k,δ,x_1} (see (7.14)), we see that (7.18) implies that $\Phi_{k,\delta}(x_0) \neq \Phi_{k,\delta}(y_0)$. The lemma follows. $\square$

Let s be a local trivializing section of L on an open set $D \subset X$. As before, we fix $p \in D$ and let $x = (x_1, \ldots, x_{2n-1})$, $z_j = x_{2j-1} + ix_{2j}$, $j = 1, \ldots, n-1$, be local coordinates of X defined in some small neighbourhood of p such that (4.31) hold. We may assume that the local coordinates x defined on D. Take $m > N_\ell$ be a large constant and let $u_{m,\delta,p}$ be as in (7.6). On D, we write $u_{m,\delta,p} = U_{k,s}\widetilde{u}_{m,\delta,p}$, $\widetilde{u}_{m,\delta,p} \in \mathscr{C}^\ell(D)$. Put $D_{p,m} := \Big\{x = (x_1, \ldots, x_{2n-1});\ |x| < \frac{1}{m \log m}\Big\}$. We need the following.

Lemma 7.5. *With the notations above, there exists $m_0 > 0$ such that $r_k m_0^{1/3} > 4$ and for all $m \geq m_0$ and $p \in X$,*

$$\inf\left\{|u_{m,\delta,p}(x)|_{h^m}\,;\, x \in D_{p,m}\right\} \geq \frac{1}{8}\delta c_p, \tag{7.19}$$

where $c_p = \frac{1}{2}\pi^{-n}\left|\det R^L_p\right| \int \chi(y_{2n-1})\chi(y_1)\cdots\chi(y_{2n-2})dy$, and for every $q \in X$ with $\operatorname{dist}(q, x) \geq \frac{r_k}{4}$, for all $x \in D_{p,m}$, we have

$$|u_{m,\delta,p}(q)|_{h^m} \leq \frac{1}{2}\inf\left\{|u_{m,\delta,p}(x)|_{h^m}\,;\, x \in D_{p,m}\right\}, \tag{7.20}$$

where $r_k > 0$ is as in Theorem 7.4.

Proof. Let $m > N_\ell$ be large enough so that

$$r_k m^{1/3} > 4. \tag{7.21}$$

As in (7.7), we have $\bmod O(m^{-\infty})$ in $\mathscr{C}^\ell(D)$

$$\widetilde{u}_{m,\delta,p}(x) \equiv \int e^{im\psi_0(x,F_m^*y,u)} m^{-n} a_\delta(x, F_m^*y, u, m)\chi(y_{2n-1})\chi(y_1)\ldots\chi(y_{2n-2})dudy. \tag{7.22}$$

From (7.22), we can repeat the proof of the first formula of (7.9) with minor changes and get (7.19). We only need to prove (7.20). Let $q \in X$ with $\operatorname{dist}(q, x) \geq \frac{r_k}{4}$, for all $x \in D_{p,m}$. If $q \notin D$, from (i) in Theorem 1.2, we can check that $|u_{m,\delta,p}(q)|_{h^m} = O(m^{-\infty})$.

We may thus assume that $q \in D$. For simplicity, we may suppose that $\operatorname{dist}(x_1, x_2) = |x_1 - x_2|$ on D. We write $q = (q_1, \ldots, q_{2n-1})$. Since $\operatorname{dist}(q, x) \geq \frac{r_k}{4}$, for all $x \in D_{p,m}$, from (7.21), we have $|q| \geq \frac{1}{4m^{1/3}}$ for m large. Thus, $|q'| \geq \frac{1}{8m^{1/3}\log m}$ or $|q_{2n-1}| \geq \frac{1}{8m^{1/3}}$, where $q' = (q_1, \ldots, q_{2n-2})$. If $|q'| \geq \frac{1}{8m^{1/3}\log m}$, by using the fact that

$$m \operatorname{Im}\psi_0(q, F_m^*y, u) \geq cm^{1/3}\frac{1}{(\log m)^2}, \quad \forall y \in \operatorname{Supp}\chi(y_{2n-1})\chi(y_1)\ldots\chi(y_{2n-2}),$$

where $c > 0$ is a constant independent of m, we conclude that

$$|\widetilde{u}_{m,\delta,p}(q)| = O(m^{-\infty}), \text{ if } |q'| \geq \frac{1}{8m^{1/3}\log m}. \tag{7.23}$$

If $|q_{2n-1}| \geq \frac{1}{8m^{1/3}}$ and $|q'| < \frac{1}{8m^{1/3}\log m}$, from (4.33), we can integrate by parts with respect to u several times and conclude that

$$|\widetilde{u}_{m,\delta,p}(q)| = O(m^{-\infty}), \text{ if } |q_{2n-1}| \geq \frac{1}{8m^{1/3}\log m} \text{ and } |q'| < \frac{1}{8m^{1/3}\log m}. \tag{7.24}$$

From (7.23) and (7.24), (7.20) follows. □

Now, we fix $m \geq N_\ell + m_0$, where m_0 is as Lemma 7.5. From Lemma 7.5, we see that we can find $x_1 \in X, x_2 \in X, \ldots, x_{d_m} \in X$ such that $X = \bigcup_{j=1}^{d_m} U_{x_j,m}$, where for each j, $U_{x_j,m}$ is an open neighbourhood of x_j with $\operatorname{Sup}\{\operatorname{dist}(q_1, q_2);\, q_1, q_2 \in U_{x_j,m}\} < \frac{r_k}{4}$, and for each j, we can find a $\mathscr{C}^\ell$ global CR section u_{m,δ,x_j} such that

$$\inf\left\{\left|u_{m,\delta,x_j}(x)\right|_{h^m};\, x \in U_{x_j,m}\right\} > 0, \tag{7.25}$$

and for every $q \in X$ with $\operatorname{dist}(q, x) \geq \frac{r_k}{4}$, for all $x \in U_{x_j,m}$, we have

$$\left|u_{m,\delta,x_j}(q)\right|_{h^m} \leq \frac{1}{2}\inf\left\{\left|u_{m,\delta,x_j}(x)\right|_{h^m};\, x \in U_{x_j,m}\right\}, \tag{7.26}$$

where $r_k > 0$ is as in Theorem 7.4. Consider the map:

$$\Psi_{m,\delta} : X \to \mathbb{CP}^{d_m-1}, \quad x \longmapsto [u_{m,\delta,x_1}, u_{m,\delta,x_2}, \ldots, u_{m,\delta,x_{d_m}}](x). \tag{7.27}$$

Let $q \in X$. Then, $q \in U_{x_j,m}$ for some $j = 1, 2, \ldots, d_m$. In view of (7.25), we see that $u_{m,\delta,x_j}(q) \neq 0$. Thus, $\Psi_{m,\delta}$ is well-defined as a smooth map.

Theorem 7.6. *The map* $(\Phi_{k,\delta}, \Psi_{m,\delta}) : X \to \mathbb{CP}^{(n+1)d_k-1} \times \mathbb{CP}^{d_m-1}$ *is a* $\mathscr{C}^\ell$ *CR embedding, where* $\Phi_{k,\delta}$ *is given by* (7.16)

Proof. In view of Theorem 7.4, we only need to show that $(\Phi_{k,\delta}, \Psi_{m,\delta})$ is injective. Let $q_1, q_2 \in X$, $q_1 \neq q_2$. Assume first that $\operatorname{dist}(q_1, q_2) \leq \frac{r_k}{4}$. From Theorem 7.4, we know that $\Phi_{k,\delta}(q_1) \neq \Phi_{k,\delta}(q_2)$ and hence $(\Phi_{k,\delta}(q_1), \Psi_{m,\delta}(q_1)) \neq (\Phi_{k,\delta}(q_2), \Psi_{m,\delta}(q_2))$. We assume that $\operatorname{dist}(q_1, q_2) > \frac{r_k}{4}$. From (7.26), it is straightforward to check that $\Psi_{m,\delta}(q_1) \neq \Psi_{m,\delta}(q_2)$ and thus $(\Phi_{k,\delta}(q_1), \Psi_{m,\delta}(q_1)) \neq (\Phi_{k,\delta}(q_2), \Psi_{m,\delta}(q_2))$. The theorem follows. □

Proof of Theorem 1.4. With the notations above, consider the Segre embedding:

$$\Upsilon : \mathbb{CP}^{(n+1)d_k-1} \times \mathbb{CP}^{d_m-1} \to \mathbb{CP}^{(n+1)d_kd_m-1},$$
$$([z_1, \ldots, z_{(n+1)d_k}], [w_1, \ldots, w_{d_m}]) \to [z_1w_1, z_1w_2, \ldots, z_1w_{d_m}, z_2w_1, \ldots, z_{(n+1)d_k}w_{d_m}].$$

It is easy to see that Υ is a smooth holomorphic embedding. From this observation and Theorem 7.6, we conclude that

$$\Upsilon \circ (\Phi_{k,\delta}, \Psi_{m,\delta}) : X \to \mathbb{CP}^{(n+1)d_kd_m-1}$$

is a $\mathscr{C}^\ell$ CR embedding. We have proved that for every $M \geq k + N_\ell + m_0$, we can find CR sections $s_0 \in \mathscr{C}^\ell(X, L^M), s_1 \in \mathscr{C}^\ell(X, L^M), \ldots, s_{d_M} \in \mathscr{C}^\ell(X, L^M)$, such that the map $x \in X \to [s_0(x), s_1(x), \ldots, s_{d_M}(x)] \in \mathbb{CP}^{d_M}$ is an embedding. Theorem 1.4 follows. □

Let us finally mention that a projective CR manifold admits Lefschetz pencil structures of degree k, for any k large enough, cf. [21].

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